Scattering and Absorption by Rotating Black Holes

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Overview

1. Introduction to Black Hole Perturbations
   • Motivations + Assumptions
   • Geodesics

2. Waves on the Schwarzschild space-time
   • Wave equation
   • Effective Potential

3. Wave Dynamics
   • Wavepacket scattering
   • Quasi-normal modes

4. Time-independent scattering
   • Partial wave analysis
   • Cross Sections

5. Scattering by rotating black holes
   • Superradiance
   • Polarization
Black hole perturbations

What do I mean by a black hole perturbation?

- A classical black hole solution $g_{\mu\nu}$
  + a perturbing field (e.g. $\Phi$ or $\Psi$ or $F_{\mu\nu}$ or $\epsilon h_{\mu\nu}$).

Assumptions:

- **Weak** $\Rightarrow$ negligible back-reaction on metric.
- **Linear** or linearized perturbation $\Rightarrow$ superposition.
- **Massless** (usually).
- Physically-motivated boundary conditions.
- Perturbations may be **classical** or quantum (i.e. 1st or 2nd quantized).
Black Hole Solutions

“Black holes have no hair”. In classical 4D GR, black holes are described by just three parameters.

- Mass $M$
- Charge $Q$
- Angular Momentum $J$.

Classification:

- Schwarzschild ($Q = 0, J = 0$).
- Reissner-Nordström ($Q \neq 0, J = 0$).
- Kerr ($Q = 0, J \neq 0$).
- Kerr-Newman ($Q \neq 0, J \neq 0$).
Fields have spin $s$ and, maybe, rest mass $m$.

- $s = 0$. Scalar field. *Klein-Gordon* eqn. Pion $\pi^0$.
- $s = \frac{1}{2}$. Spinor field. *Dirac* eqn. Neutrino $\nu$.
- $s = 2$. Tensor field. ‘Linearized’ Grav. Perturbations. Graviton (?).
Motivations (I): Black Hole Stability

Q. Why study black hole perturbations?

A1. To examine BH stability.

A black hole is stable if all (physically-reasonable) perturbations in (classical, linearized) fields are bounded in time.

Classical BH solutions are generally stable, but there are some interesting special cases:

- Superradiant instability (fast-rotating BHs)
- Gregory-Laflamme instability (higher-dimensional black objects)
Motivations (II): Gravitational Wave Detection

A2. *Perturbed black holes are sources of gravitational waves.*

**Gravitational Waves** are a key prediction of General Relativity
- Very weak ($h \sim 10^{-21}$). Yet to be detected!
- Weakly-interacting, coupled only to bulk motion of matter.

GWs will carry **strong signals from black holes** in process of:
- **Formation:** gravitational collapse and supernovae.
- **Merger:** Pair of solar-mass BHs in binary system.
- **Inspiral:** Solar-mass BHs in orbit around supermassive BHs (“radiation reaction” problem).
Motivations (II): Gravitational Wave Detection

• Precise modelling of BH signals requires full non-linear numerical solutions to Einstein’s field equations, but ...

• A surprising level of accuracy can be obtained in the linearized approximation:

\[ g_{\mu\nu} = g^{(0)}_{\mu\nu} + \epsilon h_{\mu\nu} \]

and ...

• A surprising amount can be learned by just studying a ‘toy model’: e.g. the massless scalar field \( \Phi \).
Motivations (III): “Quantum Gravity”

A3. To combine GR with Quantum Mechanics.

- Classically, black holes absorb and scatter radiation.
- Classical GR + Quantized fields ⇒ Hawking radiation.
- Thermal emission spectrum $T_H = 1/8\pi M$ ⇒ BH entropy $S \sim A$.
- Information loss puzzle: is the evolution of the wavefunction of the universe unitary?
- Semi-classical approach raises questions for Quantum Gravity: e.g. string theory or LQG.
Motivations (IV): Speculations and Extensions

- Analogue (“dumb”) black holes created in laboratory.

- “Higher-dimensional” black objects (BHs, strings, branes). Experimental signature at LHC?
Classical GR: Geodesics

- Free particles follow geodesics in space-time $x^\mu(\lambda)$.
- Geodesics are the generalisation of the Euclidean idea of a straight line.
- Straight line: shortest distance between two points.
- Geodesic: space-time path $x^\mu(\lambda)$ between two points along which the space-time interval is extremal.

⇒ Action principle:

$$S = \int ds = \int L d\lambda \quad \text{where} \quad L = \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}$$

where $\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda}$. 
The Schwarzschild Space-time

- **Unique** asymptotically flat space-time exterior to a spherically-symmetric grav. source (e.g. the Sun).
- In **Schwarzschild coordinates**
  \[ ds^2 = (1 - 2M/r)dt^2 - (1 - 2M/r)^{-1} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2). \]
- Units: \( G = c = 1 \), so \( M \equiv GM/c^2 \).
- Event horizon at \( r = 2M \).
- Compact objects that lie entirely within their horizon are black holes.
- Other coordinate systems may be used.
Schwarzschild Geodesics (I)

\[ ds^2 = (1 - 2M/r)dt^2 - (1 - 2M/r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \]

- Metric is independent of \( t \) and \( \phi \) ⇒ conserved quantities
- In equatorial plane (\( \theta = \pi/2, \dot{\theta} = 0 \)):

\[
(1 - 2M/r)\dot{t} = k,
\]
\[
r^2 \dot{\phi} = h.
\]

- ‘Energy’ \( k \) and ‘Angular momentum’ \( h \).
- To find an equation for \( \dot{r} \), use

\[ g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \epsilon^2 \equiv \begin{cases} 0 & \text{null} \\ 1 & \text{time-like} \end{cases} \]
Schwarzschild Geodesics (II)

- Use $g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \epsilon^2$ to get

\[
(1 - 2M/r) \dot{t}^2 - \left(1 - 2M/r\right)^{-1} \dot{r}^2 - r^2 \dot{\phi}^2 = \epsilon^2
\]

- Insert constants of motion to get energy equation:

\[
\dot{r}^2 + V_{\text{eff}}(r) = k^2 - \epsilon^2,
\]

with an effective potential

\[
V_{\text{eff}}(r) = -\frac{2M\epsilon^2}{r} + \frac{\hbar^2}{r^2} \left(1 - \frac{2M}{r}\right).
\]
**Figure:** Effective potential for a range of angular momenta.
Scattering and Absorption (I)

- Divide energy equation by $\dot{\phi}^2$ to get orbit equation

\[
\left( \frac{du}{d\phi} \right)^2 + u^2 = \frac{k^2 - \epsilon^2}{h^2} + \frac{2M\epsilon^2}{h^2}u + 2Mu^3
\]

where $u = 1/r$.

- Differentiate to get GR version of Binet’s equation

\[
\frac{d^2u}{d\phi^2} + u = \frac{M\epsilon}{h^2} + 3Mu^2.
\]

- Solutions $\phi(u)$ can be expressed in terms of elliptic functions.
Photon geodesics around a Schwarzschild black hole
Deflection-angle Approximations

For null geodesics:

- **Weak-field** deflection

\[ \Delta \theta \approx \frac{4M}{b} \]

\[ \Rightarrow \lim_{\theta \to 0} \frac{d\sigma}{d\Omega} \sim \frac{1}{\theta^4} \]

- **Spiral-scattering** deflection: For geodesics passing close to the **unstable orbit** at \( r = 3M \)

\[ \Delta \theta \sim - \ln \left[ \frac{(b - b_c)}{3.48M} \right] \]
Absorption

- Critical impact parameter \( b_c = 3\sqrt{3}M \) (massless)
  - \( b > b_c \) : scattered.
  - \( b < b_c \) : absorbed.

- Absorption cross section:
  \[
  \sigma_a = \pi b_c^2 = 27\pi M^2
  \]
Wavefront Scattering Simulation

Figure: Simulation by Kirill Ignatiev, UCD, 2008.
Waves on BH space-times

- Look for maximum physical insight, minimum mathematics!
- Assume weak (no back-reaction), minimally-coupled and classical fields.
- Scalar field $\Phi$: ‘toy model’ for gravitational radiation $h_{\mu\nu}$.
- Examine evolution on Schwarzschild space-time.
Scalar Wave Equation

- Wave equation:

\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} \left( \sqrt{-g} g^{\mu \nu} \frac{\partial \Phi}{\partial x^\nu} \right) + m^2 \Phi = 0.
\]

- Linearity \(\Rightarrow\) Superposition.

- Spherical-symmetry \(\Rightarrow\) Separation of variables

\[
\Phi_{lm}(t, r, \theta, \phi) = \frac{u_l(r)}{r} Y_{lm}(\theta, \phi) e^{-i\omega t},
\]

- Radial equation:

\[
\left(1 - \frac{2M}{r}\right) \frac{d}{dr} \left[ \left(1 - \frac{2M}{r}\right) \frac{du_l}{dr} \right] + \left[\omega^2 - V_l(r)\right] u_l = 0,
\]
Tortoise Coordinate

- Introduce a tortoise coordinate $r_*$,

$$\frac{dr_*}{dr} = \left( 1 - \frac{2M}{r} \right) \Rightarrow r_* = r + 2M \ln \left| \frac{r}{2M} - 1 \right|$$

- Moves the horizon to $r_* = -\infty$. 
Effective Potential

- Radial equation

\[ \frac{d^2 u_l}{dr^2} + \left[ \omega^2 - V_l(r) \right] u_l = 0, \]

with effective potential

\[ V_l(r) = \left( 1 - \frac{2M}{r} \right) \left( \frac{l(l+1)}{r^2} + \frac{2M}{r^3} + m^2 \right). \]
Effective Potential

Massless waves ($m = 0$)

\[ V_{\text{eff}} / M^{-1} \]

\( r^* \)

\( M = 10 \)

\( l = 40 \)

\( l = 70 \)
Boundary Conditions

- $V_{\text{eff}}(r_*)$ tends to constant limits at infinity and the horizon ($r_* \rightarrow \pm \infty$).

- Two independent solutions at each limit:

$$u_i(r) \sim \begin{cases} 
\exp(\pm i\omega r_*), & r_* \rightarrow -\infty, \\
\exp(\pm ipr_*), & r_* \rightarrow +\infty,
\end{cases} \quad (1)$$

where $p = (\omega^2 - m^2)^{1/2}$.
Field Current

- Wave equation ⇒

\[ \Phi^* g^{\mu \nu} \Phi_{;\mu \nu} - \Phi g^{\mu \nu} \Phi^*_{;\mu \nu} = 0. \]

- Define a conserved current \( J_\mu \):

\[ J^{\mu}_{;\mu} = (-g)^{-1/2} \partial_{\mu} \left[ (-g)^{1/2} g^{\mu \nu} J_\nu \right] = 0, \]

- To find the probability density as perceived by a specific observer, we take the contraction of \( J_\mu \) and the observer’s world line \( \dot{x}^\mu \):

\[ \rho c^2 = \dot{x}^\mu J_\mu \]
‘Probability Density’

- An observer at fixed $r, \theta, \phi \Rightarrow \dot{x}^\mu = [(1 - 2m/r)^{-1/2}, 0, 0, 0]$
- Measures time-like component

$$\rho c^2 = (1 - 2M/r)^{-1/2}\omega,$$

Diverges as $r \to 2M$. (Schwarzschild t oordinate singularity)

- But an infalling observer would measure something different: $\dot{x}^\mu = [(1 - 2M/r)^{-1}, -\sqrt{2M/r}, 0, 0]$

$$\dot{x}^\mu J_\mu = \rho c^2 \sim (1 - 2M/r)^{-1}\omega \left(1 \pm \sqrt{\frac{2M}{r}}\right)$$

$$\sim \omega \left(1 \mp \sqrt{\frac{2M}{r}}\right)^{-1}$$
Infalling observer $\dot{x}^\mu = [(1 - 2M/r)^{-1}, -\sqrt{2M/r}, 0, 0]$ measures a time-like component

$$\dot{x}^\mu J_\mu \sim \omega \left(1 \mp \sqrt{\frac{2M}{r}}\right)^{-1}$$ as $r \to 2M$

At $r = 2M$, this is regular for ingoing $e^{-i\omega r_*}$ solution ...  
... but divergent for outgoing solution $e^{i\omega r_*}$.  
⇒ Apply ingoing boundary condition at $r = 2M$.  
Horizon acts like a one-way membrane.
Wavepacket Scattering

- What happens when a black hole is perturbed slightly?
- Try firing a massless Gaussian wavepacket at a black hole [Vishveshwara, Nature, 1970].
  - Pick a specific $l$ mode
  - Numerically solve 1+1 PDE wave equation for $\phi_l(t, r_*)$:
    \[
    \frac{\partial^2 \phi_l}{\partial t^2} - \frac{\partial^2 \phi_l}{\partial r_*^2} + V_l(r_*)\phi_l = 0
    \]
  - Initial condition $\phi_l(0, r_*)$ and $\partial_t \phi_l(0, r_*) = -v \phi_l(0, r_*)$.
  - Use finite difference method (e.g. Leapfrog method).
  - Apply ingoing boundary condition as $r_* \to -\infty$.
- Try various wavepacket widths and speeds
Wavepacket Scattering (II)

“Halfway through the defence of my Ph.D, the examiner from the mathematics department asked the question: why should one bother to prove the stability of an object that was impossible to observe and was of doubtful existence in the first place? My thesis advisor did not like the question in the least, and the rest of the examination ended up as a verbal battle between the two which I watched with great satisfaction. But the question remained: how do you observe a solitary black hole? To me the answer seemed obvious. It had to be through the scattering of radiation, as the black hole left its fingerprint on the scattered wave.”

Wavepacket Scattering (III)

Figure: Vishveshwara’s scattering simulation
Figure: The time-dependent response to wavepacket scattering at fixed position far from the hole. This figure is taken from Andersson & Jensen (2001) [gr-qc/0011025].
Black Hole Response

Response of the black hole undergoes three distinct stages:

- Initial back-scattering
- ‘Quasi-normal mode’ ringing
- Power-law decay

Observable implications:

- Ringing and decay depend on black hole parameters not initial perturbation
- Quasi-normal modes (QNMs) ⇔ unstable orbit ⇔ peak in effective potential.
- QNMs ⇒ black holes not neutron stars.
- QNM frequencies and decay rates are a distinctive BH signature.
Quasi-Normal Modes

What are QNMs and how do we find their frequencies?

Defined by boundary conditions:

- **Ingoing** at horizon: \( u_I \sim e^{-i\omega r_*} \) as \( r_* \to -\infty \).
- **Outgoing** at infinity: \( u_I \sim A_{out} e^{+i\omega r_*} \) as \( r_* \to +\infty \).
- Two boundary conditions \( \Rightarrow \) Discrete spectrum \( \omega_q \).
- **Complex** \( \omega_q \):
  - Frequency: \( \text{Re}(\omega_q) \sim 1/T_{\text{period}} \)
  - Decay rate: \( -\text{Im}(\omega_q) \sim 1/\tau_{\text{life}} \)
"Any initial perturbation will, during its last stages, decay in a manner characteristic of the black hole and independent of the original cause. In other words, during the last stages, the black hole will emit gravitational waves with frequencies and rates of damping characteristic of itself, in the manner of a bell sounding its last dying pure notes."

QNM Frequencies

Figure: *The Quasi-Normal Mode Spectrum*. Plot taken from Fig. 1, E.W. Leaver, *Proc. R. Soc. Lond. A* 402, 285–298 (1985). It shows the QNM frequencies of the grav. field of the Schw. BH., for \( l = 2 \) and \( l = 3 \) modes. With our conventions, the y-axis should read \(-\text{Im}(\omega)\).
Green’s Function Analysis

- Consider time evolution of field as initial value problem
- Field expressed in terms of a Green’s function \( G(r_*, y, t) \)

\[
\Phi_l(r_*, t) = \int G(r_*, y, t) \partial_t \Phi_l(y, 0) dy + \int \partial_t G(r_*, y, t) \Phi_l(y, 0) dy.
\]

- The retarded Green’s function is defined by

\[
\left[ \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V(r) \right] G(r_*, y, t) = \delta(t) \delta(r_* - y),
\]

and the condition \( G(r_*, y, t) = 0 \) for \( t \leq 0 \).
• G is ingoing at horizon:

\[
\frac{\partial G}{\partial r_*} + i\omega G = 0, \quad r \to 2M
\]

• and outgoing at infinity:

\[
\frac{\partial G}{\partial r_*} - i\omega G = 0, \quad r \to \infty.
\]

• Take Fourier transform of G:

\[
\hat{G}(r_*, y, \omega) = \int_{0^-}^{\infty} G(r_*, y, t)e^{i\omega t} dt
\]

• The inversion formula is

\[
G(r_*, y, t) = \frac{1}{2\pi} \int_{-\infty + ic}^{\infty + ic} \hat{G}(r_*, y, \omega)e^{-i\omega t} d\omega
\]
Green’s function

- The Green’s function can be found from any two linearly-independent solutions of wave eq.
- Use $\phi^{(\text{up})}_l$ and $\phi^{(\text{in})}_l$, with boundary conditions

\[
\phi^{(\text{in})}_l \sim \begin{cases} 
  e^{-i\omega r_*}, & r_* \to -\infty, \\
  A_{\text{in}} e^{-i\omega r_*} + A_{\text{out}} e^{+i\omega r_*}, & r_* \to +\infty,
\end{cases}
\]

- and

\[
\phi^{(\text{up})}_l \sim \begin{cases} 
  B_{\text{in}} e^{-i\omega r_*} + B_{\text{out}} e^{+i\omega r_*}, & r_* \to -\infty, \\
  e^{+i\omega r_*}, & r_* \to +\infty.
\end{cases}
\]

where $A_{\text{in}}$, $A_{\text{out}}$, $B_{\text{in}}$ and $B_{\text{out}}$ are complex constants.
Green’s function

- Green’s function is
  \[ \hat{G}(r_*, y, \omega) = -\frac{1}{W} \left\{ \begin{array}{ll}
  \phi_i^{(\text{in})}(r_*)\phi_i^{(\text{up})}(y), & r_* < y, \\
  \phi_i^{(\text{up})}(r_*)\phi_i^{(\text{in})}(y), & r_* > y
  \end{array} \right. \]

- Properties of \( \phi_i^{(\text{in})} \) and \( \phi_i^{(\text{up})} \) take care of boundary conditions.
- Here \( W \) is the Wronskian
  \[ W = \phi_i^{(\text{in})} \frac{d\phi_i^{(\text{up})}}{dr_*} - \phi_i^{(\text{up})} \frac{d\phi_i^{(\text{in})}}{dr_*} = 2i\omega A_{\text{in}}. \]

- If \( \omega \) is real, \( W \) is conserved (i.e. \( W \) is independent of \( r \)).
Contour integration

- Full GF found by integrating over frequencies:

\[ G(r_*, y, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{G}(r_*, y, \omega) e^{-i\omega t} d\omega \]

- ‘Poles’ of GF where \( A_{\text{in}} = 0 \) ⇒ QNM frequencies!
- Compute \( G \) with a contour integral:
Contour Integral

Three stages of black hole response:
- Initial back-scattering
- ‘Quasi-normal mode’ ringing
- Power-law decay

Three parts of contour integral:
- High-frequency arcs
- Poles of Green’s function (Cauchy’s theorem)
- Branch cut integral
Time-Independent Plane Wave Scattering

An infinite monochromatic plane wave impinges on an isolated black hole:
Time-Independent Plane Wave Scattering

Dimensionless Parameters:

- BH coupling: \( M \omega = \pi r_s / \lambda \)
- BH rotation: \( 0 \leq a \leq 1 \)

Physical Observables:

- \( \sigma_a \): absorption cross section.
- \( \frac{d\sigma}{d\Omega} \): differential scattering cross section.
- \( 0 < P < 1 \): partial polarisation.
Weak-Field Approximations: $\lambda \gg r_s, a = 0$

scattering cross section: $\frac{1}{M^2} \frac{d\sigma}{d\Omega}$

<table>
<thead>
<tr>
<th>Type</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>scalar</td>
<td>$\frac{1}{\sin^4(\theta/2)}$</td>
</tr>
<tr>
<td>neutrino</td>
<td>$\frac{\cos^2(\theta/2)}{\sin^4(\theta/2)}$</td>
</tr>
<tr>
<td>photon</td>
<td>$\frac{\cos^4(\theta/2)}{\sin^4(\theta/2)}$</td>
</tr>
<tr>
<td>gravitational</td>
<td>$\frac{\cos^8(\theta/2) + \sin^8(\theta/2)}{\sin^4(\theta/2)}$</td>
</tr>
</tbody>
</table>

General rule:

$$\lim_{\lambda \to \infty} \left( \frac{1}{M^2} \frac{d\sigma}{d\Omega} \right) \sim \frac{\cos^{4s}(\theta/2)}{\sin^4(\theta/2)} + \delta_{s,2} \sin^4(\theta/2)$$
Partial Wave Approach

- Perturbation theory is only appropriate when coupling is small, $M\omega \ll 1$.
- If the wavelength is similar to the horizon size, partial wave approach is best.
- A plane wave can be decomposed into partial wave modes:

$$\Phi_{\text{plane}} = e^{ipz} \sim \frac{1}{2ipr} \sum_{l=0}^{\infty} (2l+1)P_l(\cos \theta) \left[ e^{ipr} + (-1)^{l+1} e^{-ipr} \right].$$

- Asymptotically, as $r \to \infty$, solution looks like plane wave + outgoing scattered wave.
- Not quite possible because gravitational force is long-ranged $1/r$ like Coulomb force
- Asymptotic solutions $u_l \sim e^{\pm ipr}$. 
Partial Wave Analysis

- **Distorted plane wave:**

\[
\Phi_{\text{dist. plane}} = \frac{1}{2i pr} \sum_{l=0}^{\infty} (2l + 1) P_l(\cos \theta) \left[ e^{ipr*} + (-1)^{l+1} e^{-ipr*} \right].
\]

- \(e^{ipr*} \sim r^{2iMp} e^{ipr}\): extra phase factor which bends wavefronts even in the far field.

- **Construct partial wave solution:**

\[
\Phi \sim \Phi_{\text{dist. plane}} + \frac{f(\theta)}{r} e^{ipr*} = \frac{1}{r} \sum_{l=0}^{\infty} a_l(2l + 1) P_l(\cos \theta) \phi_l^{(in)}(r_\ast).
\]

- Find \(a_l\) by matching ingoing part of rhs with ingoing part of plane wave.
Phase shifts $\delta_l$

- Radial solutions with boundary conditions

$$u_l(r) \sim \phi_l^{(\text{in})} \sim \begin{cases} e^{-i\omega r_*}, & r \to 2M \\ A_{\text{in}} e^{-ip r_*} + A_{\text{out}} e^{ip r_*}, & r \to \infty \end{cases}$$

- Scattering amplitude $f$ is a partial wave series:

$$f(\theta) = \frac{1}{2i\omega} \sum_{l=0}^{\infty} (2l + 1) \left( e^{2i\delta_l} - 1 \right) P_l(\cos \theta).$$

where the phase shifts $e^{2i\delta_l}$ are defined by

$$e^{2i\delta_l} = (-1)^{l+1} \frac{A_{\text{out}}}{A_{\text{in}}}$$

- The phase shifts encode all information about the scattering.
Scattering and Absorption

- The differential scattering cross section is
  \[ \frac{d\sigma}{d\Omega} = |f(\theta)|^2 \]

- The absorption cross section is
  \[ \sigma_{\text{abs}} = \frac{\pi}{\omega^2} \sum_{l=0}^{\infty} (2l + 1) T_l \]

where the transmission factors \( T_l \) are
  \[ T_l = 1 - |e^{2i\delta_l}|^2 \]
Absorption Cross Section (I)

Figure: Absorption cross section for a massless scalar wave impinging upon a Schwarzschild black hole.
Absorption Cross Section (II)

Schwarzschild absorption cross section for massless scalar, spinor, EM and grav. waves

\[ \sigma_a / \pi M^2 \]

- \( s = 0 \)
- \( s = 1/2 \)
- \( s = 1 \)
- \( s = 2 \)

Grav. Wave
Electromagnetic
Spinor
Scalar

[E.M. data from Crispino, Oliveira, Higuchi and Matsas (2007)]
Phase shifts

- Semi-classical interpretation:

\[ b \approx (l + \frac{1}{2}) \omega \]

Figure: *Scalar wave phase shifts at \( M\omega = 2.0 \).*
Semi-classical picture: Phase shifts $\Leftrightarrow$ Geodesics
Computing Scattering Cross Sections (I)

- Scattering amplitude

\[ f(\theta) = \frac{1}{2i\omega} \sum_{l=0}^{\infty} (2l + 1) \left( e^{2i\delta_l} - 1 \right) P_l(\cos \theta). \]

- Series is poorly convergent.
- Why? Amplitude is divergent as \( \theta^{-2} \) at small angles \( \leftrightarrow \) long-ranged interaction \( \leftrightarrow \) poorly-convergent infinite series.
- Same problem for Coulomb scattering.

\[ e^{2i\delta_l^N} = \frac{\Gamma(l + 1 - i\beta)}{\Gamma(l + 1 + i\beta)} \sim \beta \ln(l) \text{ as } l \to \infty \]

where \( \beta = Z\alpha m/p \leftrightarrow GM\omega/c^3 \).
Computing Scattering Cross Sections (II)

- But Coulomb case has exact solution

\[ f_N(\theta) \equiv \frac{1}{2ip} \sum_{l=0}^{\infty} \frac{\Gamma(l + 1 - i\beta)}{\Gamma(l + 1 + i\beta)} (2l + 1) P_l(\cos \theta) \]  

\[ = \frac{\beta}{2p} \frac{\Gamma(1 - i\beta)}{\Gamma(1 + i\beta)} \left[ \sin(\theta/2) \right]^{-2 + 2i\beta}. \]  

- Cross section:

\[ \frac{d\sigma}{d\Omega} = \frac{\beta^2}{4p^2 \sin^4(\theta/2)} = \frac{M^2(1 + v^2)^2}{4v^4 \sin^4(\theta/2)}. \]  

- Idea: use this result to remove the long-range effect of Newtonian potential, leaving a convergent series.

- This method works well for scalar wave, but not for waves of higher spin.
Computing Scattering Cross Sections (III)

- **Alternative method**: Series reduction method dating from 1950s.
- Given a Legendre polynomial series

\[ f(\theta) = \sum_{l=0} a^{(0)}_l P_l(\cos \theta) \]

divergent at \( \theta = 0 \), define the \( m \)th reduced series,

\[ (1 - \cos \theta)^m f(\theta) = \sum_{l=0} a^{(m)}_l P_l(\cos \theta). \]

- The reduced series is obviously less divergent at \( \theta = 0 \) ⇔ better convergence.
- Iterative formulae:

\[ a^{(i+1)}_l = a^{(i)}_l - \frac{l + 1}{2l + 3} a^{(i)}_{l+1} - \frac{l}{2l - 1} a^{(i)}_{l-1}. \]
Scattering Cross Sections: Scalar

Figure: Scalar scattering cross sections. N.B. Log scale.
Scattering Cross Sections: Spinor

Figure: Spin-half scattering cross sections.
Glory Scattering in Optics
Glory Scattering in Optics (II)
Glory Scattering Approximation

- Semi-classical (WKB) approximation:

\[ \lim_{\theta \to \pi} \frac{d\sigma}{d\Omega} \sim 2\pi M^2 \times 4.3M\omega \times J_2|s| \times (5.3465 M\omega \sin \theta) \]

- Zero flux in backward direction for polarized fields \((s \neq 0)\).
Figure: *Gravitational wave scattering cross sections (a = 0).*
Rotating Black Holes + Grav. Waves

• How do results change when the black hole is rotating? ($a = 0.99$)

• How does rotation of BH affect polarization of gravitational wave?

• N.B. I will only consider the special case when the wave is incident along the rotation axis.
Gravitational Wave Polarizations

- **Linear** polarizations: $(\pm)$ and $(\times)$.

- **Circular** polarizations: right- and left-handed: $(\pm i(\times))$. 
Rotating BH: On-axis Incidence

Scattering cross section [Futterman, Handler & Matzner 1988] :

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 + |g(\theta)|^2$$

$$f(\theta) = \frac{\pi}{i\omega} \sum_{l,P=\pm 1} \left[ \exp(2i\delta_{l2}^P) - 1 \right] -2S^2_l(0) -2S^2_l(\theta)$$

$$g(\theta) = \frac{\pi}{i\omega} \sum_{l,P=\pm 1} P(-1)^l \left[ \exp(2i\delta_{l2}^P) - 1 \right] -2S^2_l(0) -2S^2_l(\pi - \theta)$$

- $sS^m_l(\theta)$ are the spin-weighted spheroidal harmonics
- $g(\theta)$ is spin-flip amplitude (e.g. left-handed $\leftrightarrow$ right-handed.)
Parity-Dependence

- The phase shifts $\delta^P_{lm}$ are parity-dependent:

$$\exp(2i\delta^+_P) = \frac{\text{Re}(C) + 12iM\omega}{\text{Re}(C) - 12iM\omega} \exp(2i\delta^-_{lm})$$

$$|\text{Re}(C)|^2 = \left((\lambda + 2)^2 + 4\omega m - 4a^2\omega^2\right)\left(\lambda^2 + 36\omega m - 36a^2\omega^2\right) + (2\lambda + 3)(96a^2\omega^2 - 48\omega m) - 144a^2\omega^2$$

- Hence $g \neq 0$ and helicity is not conserved!

- $\Rightarrow$ Non-zero flux in backward-scattering direction. For $a = 0$,

$$\lim_{M\omega \to 0} \frac{d\sigma}{d\Omega}(\theta = \pi) = |g(\theta)|^2 = M^2$$
Computing Phase Shifts

Numerical Method:

- Solve radial Teukolsky equation with ingoing boundary condition at (outer) horizon, $R(r) \sim |r - r_+|^{-i\omega r}$.

- ‘Peeling’ behaviour at infinity ($R_{out} \sim r^3 e^{i\omega r^*}$ and $R_{in} \sim r^{-1} e^{-i\omega r^*}$) makes things numerically awkward.

- Use Sasaki and Nakamura transformation, $\chi(R, R')$, so that $\chi_{out} \sim e^{i\omega r^*}$ and $\chi_{in} \sim e^{-i\omega r^*}$

- Start with a Taylor series expansion to 5th order in $|r - r_+|$, numerically integrate outwards, and match onto expansions at $r \to \infty$ to 10th order in $1/r$. 
Calculating Spin-Weighted Spheroidal Harmonics

Method:

- Expand spheroidal harmonics $-2S_l^2$ in spherical harmonics $-2Y_l^m$:

$$-2S_l^m(\theta; a\omega) = \sum_j b_{lj} -2Y_j^m(\theta)$$

- Solve eigenvalue equation: $A \mathbf{b} = -E_{lm} \mathbf{b}$ to find angular separation constant.

- $\mathbf{A}$ is band-diagonal $\Rightarrow$ fast method

- Compute eigenfunctions using recurrence relations [S.A. Hughes (2000)].
Kerr Absorption Cross Section

- Assume incidence along the rotation axis.
- The absorption cross section is found from a sum over modes:

\[ \sigma_a = \frac{4\pi^2}{\omega^2} \sum_{l=2}^{\infty} \left| -2 S_l^2(0) \right|^2 \left( 1 - \left| e^{2i\delta_{l\pm}} \right|^2 \right) \]

- Circular polarization of incident wave is either co-rotating (\( \omega > 0 \)) or counter-rotating (\( \omega < 0 \)) with the black hole.
Kerr Absorption Cross Section

- Rotating BH $\Rightarrow$ superradiance (stimulated emission).

![Graph showing Kerr Absorption Cross Section](image)
Kerr Scattering: Long Wavelengths

$m|\omega| = 0.2$

$m|\omega| = 0.4$

$m|\omega| = 0.6$

$m|\omega| = 0.8$
Kerr Scattering: Long Wavelength

- Co-rotating incident wave $\Rightarrow$ Enhanced helicity-reversed back-scattering caused by superradiance

![Graph showing scattering cross section vs. scattering angle for different values of $a$ and $\omega$.]
Kerr Scattering: Short Wavelengths

- $M|\omega| = 1.0$
- $M|\omega| = 2.0$
- $M|\omega| = 3.0$
- $M|\omega| = 4.0$

Scattering angle (deg)
Conclusions

- BH grav. wave scattering violates helicity conservation.
- Long-range force $\Rightarrow$ divergence on-axis $d\sigma/d\Omega \sim 1/\theta^4$
- Unstable orbits $\Rightarrow$ spiral scattering oscillations.
- BH rotation $\Rightarrow$ enhanced back-scattering ($\sim \times 20$).

Future work: General off-axis scattering. Diffraction patterns from higher-dimensional BHs? Analogue BHs?
References:


