Ergoregion instability rules out black hole doubles

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I. INTRODUCTION

Black holes (BHs) in Einstein-Maxwell theory are characterized by three parameters [1]: Mass $M$, electric charge $Q$ and angular momentum $J \equiv aM \leq M^2$. BHs are thought to be abundant objects in the Universe [2]. Their mass is estimated to vary between $3M_\odot$ and $10^{9.5}M_\odot$ or higher. They are likely to be electrically neutral because of the effect of surrounding plasma [3] and their angular momentum is expected to be close to the extremal limit because of accretion and merger events [4, 5]. An example of astrophysical BH is the compact primary of the binary X-ray source GRS 1915 + 105, which recent observations identify as a rapidly-rotating object of spin $a \gtrsim 0.98 M \odot$ [6]. Many of the supermassive BHs which are thought to power quasars seem to be rotating near the Kerr bound [2].

Despite the wealth of circumstantial evidence, there is no definite observational proof of the existence of astrophysical BHs. (A review and a critique of current evidence can be found in Ref. [2] and Ref. [8], respectively. See also Ref. [9] for a stimulating mini-review.) Astrophysical objects without event horizon, yet observationally indistinguishable from BHs, cannot be excluded a priori. These “BH doubles” include gravastars, boson stars, wormholes and superspinars.

Gravastars. Dark energy stars or “gravastars” are compact objects with de Sitter interior and Schwarzschild exterior [10, 11]. These two regions are glued together around the would-be horizon by an ultra-stiff thin shell. In this model, a gravitationally collapsing star undergoes a phase transition that prevents further collapse. The thickness of the shell sets an upper limit to the mass of the gravastar [11, 12, 13]. (A thorough analysis of the maximum compactness of gravastars can be found in Ref. [14].) Generalizations of the original model use a Born-Infeld phantom field [15], dark energy equation of state [16] or non-linear electrodynamics [17]. Models without shells or discontinuities have been investigated in Ref. [18].

Boson stars. Boson stars are macroscopic quantum states which are prevented from undergoing complete gravitational collapse by Heisenberg uncertainty principle [19, 20]. Their models differ in the scalar self-interaction potential [21] and can be divided in three classes [22].

Miniboson stars. If the scalar field is non-interacting, the maximum boson star mass is $M_{\text{max}} \sim 0.633 m_{\text{Planck}}^2/m$ [19]. This value is much smaller than the Chandrasekhar mass for fermion stars, $M_{\text{Ch}} \sim m_{\text{Planck}}^2/m^2$. Stability of supermassive objects requires an ultralight boson of mass $m = 8.45 \times 10^{-26}\text{GeV} \left(10^6 M_\odot/M_{\text{max}}\right)$.

Massive boson stars. The requirement of ultralight bosons can be lifted if the scalar field possesses a quartic

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self-interaction potential of the form $\lambda|\phi|^4/4$. As long as the coupling constant $\lambda$ is much larger than $(m/m_{\text{Planck}})^2$, the maximum boson star mass can be of the order of the Chandrasekhar mass or larger, $M_{\text{max}} \sim 0.062\lambda^{1/2}m_{\text{Planck}}^3/m^2$. Thus supermassive objects may exist. Boson mass and coupling constant are related by $m = 3.2 \times 10^{-4} \text{ GeV} \lambda^{1/4}(10^6M_\odot/M_{\text{max}})^{1/2}$.

Nontopological soliton stars. If the self-interaction takes the form $U = m^2|\phi|^2(1 - |\phi|^2/\phi_0^2)^2$, compact non-dispersive solutions with a finite mass may exist even in the absence of gravity [24]. The critical mass of these objects is $M_{\text{max}} \sim 0.0198 m_{\text{Planck}}^2/(m\phi_0^2)$. If $\phi_0 \sim m$, a star of mass $M \sim 10^6M_\odot$ corresponds to a heavy boson of mass $m \sim 500$ GeV.

Boson stars are promising candidates for supermassive BH doubles. They are indistinguishable from BHs in the Newtonian regime. Since they are very compact, deviations in the properties of orbiting objects occur close to the Schwarzschild radius and are not easily detectable electromagnetically [25, 26]. If the scalar field interacts only gravitationally with matter, compact objects may safely inspiral “inside” the boson star, the only difference with a BH being the absence of an event horizon [27]. Lack of strong constraints on boson masses makes these models difficult to rule out.

Wormholes and other Kerr-like objects. Wormholes can be objects even simpler than BHs [28, 29, 30]. Although there is no observational evidence supporting their existence, no theoretical arguments rule them out. Wormholes which are infinitesimal variations of the Schwarzschild spacetime may be indistinguishable from Schwarzschild BHs [31]. In a string theory context, the fuzzball model replaces BHs by horizonless structures [32]. The BH geometry emerges in a coarse-grained description which “averages” over horizonless geometries. This produces an effective horizon at a radius where the individual microstate geometries start to differ appreciably.

Superspinars. High-energy effects could lead to a violation of the Kerr bound, i.e. $a \geq M$ [33]. (A large class of BHs in string theory-inspired models violates this bound.) This would imply the existence of rotating exotic objects resembling Kerr BHs. These “superspinars” are expected to have compactness of the order of extremal rotating Kerr black holes, $M \sim R$, and to exist in any mass range.

All the models listed above provide viable alternatives to astrophysical BHs. To ascertain the true nature of ultra-compact objects it is thus important to devise observational tests to distinguish BH doubles from ordinary BHs. The traditional way to distinguish a BH from a neutron star is to measure its mass. If the latter is larger than the Chandrasekhar limit, the object is believed to be a BH. However, this method cannot be used for BH doubles because of their broad mass spectrum. A possibility is to look for observables related to the accretion mechanism. For example, the luminosity of quiescent BHs is lower than the maximum luminosity which is allowed by the gas present in their environment [34]. If the BH accretion rate is much smaller than the Eddington rate, the radiative efficiency is also very small [35]. Another possibility is to exploit the absence of a boundary layer at the surface. Compact stars with accretion disks have typically a narrow viscous boundary layer near their surface, which allows the release of a considerable amount of heat energy. On the other hand, if the central object is a BH, no boundary layer is formed. Arguments of this kind have already excluded many gravastar candidates [36]. Absence of type I X-ray bursts is another powerful indicator of the presence of a BH. Several studies on type I bursts show that they are produced when gas accretes on the surface of a NS [37], which then undergoes a semi-regular series of thermonuclear explosions. Since BHs do not have surfaces, the surrounding gas cannot accumulate and thermonuclear instabilities do not develop.

Another very promising observational method to probe the structure of compact objects is gravitational wave astronomy [38]. Gravitational wave detectors such as LIGO [39], VIRGO [40], TAMA [41], or LISA [42] could provide an efficient way to study ultra-compact objects without intervening effects due to the interstellar medium. For example, the inspiral process of two compact objects allows a precise determination of their mass [43] and multipole moments [27, 44, 45, 46]. The gravitational waveform in the presence of a surface is also expected to be different than the waveform in the presence of an event horizon [47]. A first study on the distinctive features of the inspiral signal of boson stars can be found in Ref. [48]. Detection of gravitational resonant modes due to the gravitational potential well could also provide a test for the presence of a horizon [22, 50]. Preliminary studies for gravastars indicate that this method may be very efficient if the source is not too far away and gravitational wave production is significant [13, 22, 49].

In this paper, we propose a new way of discriminating BHs from ultra-compact horizonless objects. Our method uses the fact that compact rotating objects without event horizon are unstable when an ergoregion is present. The origin of this ergoregion instability can be traced back to superradiant scattering. In a scattering process, superradiance occurs when scattered waves have amplitudes larger than incident waves. This leads to extraction of energy from the scattering body [51, 52, 53]. Instability may arise whenever this process is allowed to repeat itself ad infinitum. This happens, for example, when a BH is surrounded by a “mirror” that scatters the superradiant wave back to the horizon, amplifying it at each scattering. The total extracted energy grows exponentially with time until the radiation pressure destroys the mirror in a process called BH bomb (see Refs. [54, 55]). If the mirror is inside the ergoregion, superradiance may lead to an inverted BH bomb. Part of the superradiant waves escape to infinity carrying positive
energy, causing the energy inside the ergoregion to decrease and eventually generating an instability. This may occur for any rotating star with an ergoregion: The mirror can be either its surface or, for a star made of matter non-interacting with the wave, its center. BHs are stable, which could be due to the absorption by the event horizon being larger than superradiant amplification.

The ergoregion instability appears in any system with ergoregions and no horizons [50]. (See also Ref. [57] for an exhaustive discussion.) Explicit computations for ordinary rotating stars can be found in Ref. [58, 59], where typical instability timescales are shown to be larger than the Hubble time. In this case, the ergoregion instability is too weak to produce any effect on the evolution of the star. This conclusion changes drastically for ultra-compact objects. For compactness $M \gtrsim 0.5R$ and angular momentum $J \gtrsim 0.4M^2$, we find that instability timescales range approximately from 0.1 seconds to 1 week for objects with mass in the range $M \sim 1M_\odot$ to $10^5M_\odot$, further decreasing for larger rotation rates.

Due to the difficulty of handling gravitational perturbations for rotating objects, the calculations below are mostly restricted to scalar perturbations. However, we are able to show that the equation for axial gravitational perturbations of gravastars is identical to the equation for scalar perturbations in the large $l = m$ limit, and that the timescale of gravitational perturbations is much smaller than the timescale of scalar perturbations for superspinars and generic Kerr-like objects. (This follows from a greater superradiant amplification, see Ref. [60].) Thus our investigation seems to rule out these compact, rapidly spinning objects as BH candidates.

This paper is organized as follows. In Section II we review the main characteristics of the ultra-compact objects listed above. Our discussion is non-exhaustive and strictly limited to concepts and tools which will be needed in the rest of the paper. Section II A introduces the two gravastar models which will be discussed in the subsequent analysis. Since there are no known solutions describing rotating gravastars, the formalism of Refs. [61, 62] will be used to discuss rotating gravastars. Section II B introduces boson stars [23]. Numerical results for rotating boson stars are taken from Ref. [62]. Superspinars and wormholes are introduced in Section II C. Section III presents a detailed investigation of the instability of boson stars and gravastars using the WKB approximation. The WKB analysis is then compared with full numerical results obtained by direct integration of the Klein-Gordon equation. Section IV deals with generic Kerr-like objects without horizon. Sections IV A and IV B present analytical results for two limiting cases: Slow rotation and small frequencies, and fast rotation and large frequencies, respectively. (Details of these calculations are in Appendices A and B. In Sect. V we claim that superspinars are typically unstable against various kinds of perturbations and give an explicit example: The algebraically special perturbations. Numerical results are discussed in Sect. IV C. Detectability of ergoregion instability by gravitational-wave detectors is addressed in Sect. VI. Section VII contains a brief discussion of the results and concludes the paper.

Geometrized units ($G = c = 1$) are used throughout the paper, except when numerical results for rotating boson stars from Ref. [63] are discussed (Section II B). In this case, the Newton constant is defined as $G = 0.05/(4\pi)$.

II. STRUCTURE OF BH DOUBLES

This section discusses the main properties of gravastars, boson stars and quasi-Kerr objects. The derivation of nonrotating solutions is partly based on Refs. [11, 13, 23, 31, 33, 63].

A. Gravastars

Although exact solutions for spinning gravastars are not known, they can be studied in the limit of slow rotation by perturbing the nonrotating solutions [61]. This procedure was used in Ref. [62] to study the existence of ergoregions for ordinary rotating stars with uniform density. Their analysis is repeated below for gravastars. In the following, we discuss the original thin-shell model by Mazur and Mottola [11] and the anisotropic fluid model by Chirenti and Rezzolla [13, 18].

1. Nonrotating thin-shell model

In this model, the spacetime

$$ds^2 = -f(r)dt^2 + B(r)dr^2 + r^2d\Omega_2^2$$

(2.1)

consists of three regions:

I. Interior: $0 \leq r \leq r_1$, $\rho = -p$

II. Shell: $r_1 \leq r \leq r_2$, $\rho = p$

III. Exterior: $r_2 \leq r$, $\rho = p = 0$

where $\rho$ is the energy density and $p$ is the isotropic pressure of the gravastar. In region I, $\rho = 3H_0^2/8\pi$ is constant and the metric is de Sitter:

$$f = \frac{C}{B} = C(1 - H_0^2 r^2), \quad 0 \leq r \leq r_1,$$

(2.3)

where $C$ is an integration constant to be determined from matching conditions. In region III the spacetime is described by the Schwarzschild metric,

$$f = \frac{1}{B} = 1 - \frac{2M}{r}, \quad r_2 \leq r.$$

(2.4)
In region II, the metric is determined by the system of equations,
\begin{align*}
    d\ln \rho &= \frac{dh}{1 - w - h}, \\
    d\ln h &= -\left(\frac{1 - w - h}{1 + w - 3h}\right) d\ln w,
\end{align*}
where \( w = 8\pi r^2 p \) and \( w f/r^2 = \) constant. A simple analytical solution can be obtained for a thin shell \[11\]. In the limit \( r_1 \rightarrow r_2 \), one obtains
\[ \frac{1}{B} \simeq \epsilon \frac{(1 + w)^2}{w} \ll 1, \]
where \( \epsilon \) is an integration constant. The continuity of the metric coefficients \( f \) and \( B \) at \( r_1 \) and \( r_2 \) implies that \( \epsilon \), \( C \), \( M \) and \( H_0 \) are related to \( r_1 \), \( r_2 \), \( w_1 = w(r_1) \) and \( w_2 = w(r_2) \) by \[13\]
\begin{align*}
    \epsilon &= -\ln \frac{r_2}{r_1} \left(\frac{w_2}{w_1} - \frac{1}{w_2} + \frac{1}{w_1}\right)^{-1}, \\
    C &= \left(\frac{1 + w_2}{1 + w_1}\right)^2, \\
    M &= \frac{r_2}{2} \left[1 - \frac{\epsilon(1 + w_2)^2}{w_2}\right], \\
    H_0^2 &= \frac{1}{r_1^2} \left[1 - \frac{\epsilon(1 + w_1)^2}{w_1}\right].
\end{align*}
The above relations and Eq. (2.5) completely determine the structure of the gravastar. A typical solution is shown in the upper panel of Fig. 1 for \( r_2 = 1.05 \), \( r_1 = 1 \), \( w_1 = 350 \) and \( w_2 = 1 \).

2. Nonrotating gravastars with anisotropic pressure

This model assumes a thick shell with continuous profile of anisotropic pressure to avoid the introduction of an infinitesimally thin shell. The stress-energy tensor is \( T^{\mu \nu} = \text{diag}[-\rho, p_r, p_t, p_t] \), where \( p_r \) and \( p_t \) are the radial and tangential pressures, respectively. The density function is
\[ \rho(r) = \begin{cases} 
    \rho_0, & 0 \leq r \leq r_1 \\
    \frac{2\rho_0}{(r_2 - r_1)^3}, & r_1 < r \leq r_2 \\
    0, & r_2 \leq r \leq r_3
\end{cases} \]
with boundary conditions \( \rho(0) = \rho(r_1) = \rho_0 \), \( \rho(r_2) = \rho'(r_1) = \rho'(r_2) = 0 \) and
\begin{align*}
    a &= \frac{2\rho_0}{(r_2 - r_1)^3}, \\
    b &= -\frac{3\rho_0(r_2 + r_1)}{(r_2 - r_1)^3}, \\
    c &= \frac{6\rho_0 r_1 r_2}{(r_2 - r_1)^3}, \\
    d &= \frac{\rho_0 r_2^5 - 3r_1^2}{(r_2 - r_1)^3}.
\end{align*}
The density is related to the total mass \( M \) by
\[ \frac{\rho_0}{M} = \frac{15}{2\pi(r_1 + r_2)(2r_1^2 + r_1 r_2 + 2r_2^2)}. \]
The radial pressure \( p_r \) is chosen as \[13\]
\[ p_r(\rho) = \left(\frac{\rho^2}{\rho_0}\right)^{\frac{\alpha}{1 + \alpha}} \left[1 + \frac{\rho}{\rho_0}\right]^{2}, \]
where the parameter \( \alpha \) is determined by demanding that the maximum sound speed coincides with the speed of light. (This requirement rules out superluminal behavior and implies \( \alpha \sim 2.21 \).) The metric coefficients are
\[ f = \left(1 - \frac{2M}{r_2}\right) e^{\Gamma(r) - \Gamma(r_2)}, \quad \frac{1}{B} = 1 - \frac{2m(r)}{r}, \]
where
\[ m(r) = \int_0^r 4\pi r^2 \rho dr, \]
and
\[ \Gamma(r) = \int_0^r \frac{2m(r) + 8\pi r^3 p_r}{r(r - 2m(r))} \, dr. \] (2.17)

The above equations completely determine the structure of the gravastar. Both the metric and its derivatives are continuous across \( r_2 \) and throughout the spacetime. The behavior of the metric coefficients for a typical gravastar are shown in the bottom panel of Fig. 1.

3. Slowly rotating rigid gravastars and ergoregions

There are no known solutions describing rotating gravastars. Thus an analysis of the ergoregion instability for these objects is nontrivial. Fortunately, slowly rotating solutions can be obtained using the formalism developed in [61], which we now extend to the case of anisotropic stresses.

A small amount of rotation \( \Omega \) gives corrections of order \( \Omega^2 \) in the diagonal coefficients of the metric (2.1) and introduces a non-diagonal term of order \( \Omega \),
\[ g_{t\phi} = -\zeta g_{\phi\phi}, \] (2.18)
where \( \phi \) is the azimuthal coordinate. The metric coefficient \( g_{t\phi} \) defines the angular velocity of frame dragging \( \zeta = \zeta(r) \). The full metric is
\[ ds^2 = -f(r)dt^2 + B(r)dr^2 + r^2d\theta^2 + r^2 \sin^2 \theta (d\phi - \zeta(r)dt)^2. \] (2.19)

We consider the anisotropic fluid stress-energy tensor
\[ T^{\mu\nu} = (\rho + p_t)U^{\mu}U^{\nu} + p_t g^{\mu\nu} + (p_r - p_t)s^{\mu}s^{\nu}, \] (2.20)
where
\[ U^{\mu}U_{\mu} = -1, \quad s^{\mu}s_{\mu} = 1, \quad U^{\mu}s_{\mu} = 0, \]
\[ U^t = U^\theta = 0, \quad U^\phi = \Omega U^t, \]
\[ U^t = \sqrt{-(g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi})}. \]

Equation (2.20) describes an anisotropic fluid with radial pressure \( p_r \) and tangential pressure \( p_t \), rotating with angular velocity \( \Omega \) as measured by an observer at rest in the \((t, r, \theta, \phi)\) coordinates. If the star rotates rigidly, i.e. \( \Omega = \text{constant} \), the Einstein equations at order \( \zeta \) give
\[ -8\pi \rho = \frac{B - B^2 - rB'}{r^2 B^2}, \] (2.21)
\[ 8\pi p_r = \frac{f - Bf + r f'}{r^2 B f}, \] (2.22)
\[ 8\pi p_t = -\frac{2f^2 B' + rBf'^2}{4r B^2 f^2} - \frac{f (rB'f' - 2B(f' + rf''))}{4r B^2 f^2}. \] (2.23)

An equation for \( \zeta(r) \) is obtained by considering
\[ R_{t\phi} = 8\pi \left( T_{t\phi} - \frac{1}{2} g_{t\phi} T \right). \] (2.24)

Using Eqs. (2.21) and (2.22), Eq. (2.24) is written as
\[ \zeta'' + \zeta' \left( \frac{3}{r} + \frac{j}{j} \right) = 16\pi B(r)(\zeta - \Omega)(\rho + p_t), \] (2.25)
where \( j = (fB)^{-1/2} \) is evaluated at zeroth order and \( \rho, p_t \) are given in terms of the nonrotating geometry by Eq. (2.21) and Eq. (2.22), respectively. The above equation reduces to the corresponding equation in Ref. [61] for isotropic fluids. Solutions of Eq. (2.25) describe rotating gravastars to first order in \( \Omega \).

Spinning stars may possess ergoregions. A simple but general procedure to determine their presence for slowly rotating stars is described in Ref. [62]. This method requires only a knowledge of the metric of nonrotating objects and compares favorably with more sophisticated numerical analyses [64].

The ergoregion can be found by computing the surface on which \( g_{tt} \) vanishes [62]:
\[ 0 = -f(r) + \zeta^2 r^2 \sin^2 \theta. \] (2.26)

Equation (2.26) determines the exact location of the ergoregion only when the latter is located close to the zeros of \( f(r) \). However, it may not be valid in more general situations. (See Ref. [62] for further details.) The solution of Eq. (2.26) is topologically a torus. In the equatorial plane we have
\[ r_{\zeta}(r) = \sqrt{f(r)}. \] (2.27)

The existence and the boundaries of the ergoregions can be computed from the above equations. Equation (2.26) is integrated from the origin with initial conditions \( \Omega = -\zeta' = 0 \) and \( \Omega - \zeta \) finite. Changing the value of \( \Omega - \zeta \), the whole space of slowly rotating stars can be obtained. The exterior solution satisfies \( \Omega = -\zeta = \Omega(1 - 2I/r^3) \), where \( I \) is the moment of inertia of the star. Demanding the continuity of both \( \Omega - \zeta' \) and \( \Omega - \zeta \), \( \zeta \) and \( I \) are uniquely determined. The rotation parameter \( \Omega \) depends on the initial condition at the origin.

Figure 2 shows the results for three different sequences of the gravastar models described in the previous sections. The minima of the curves are the minimum values of \( J/M^2 \) which are required for the existence of the ergoregion. Comparison with the results for stars of uniform density [62], shows that ergoregions form more easily around gravastars due to their higher compactness. Figure 2 also shows that the ergoregions spread inside the star. (The ergoregion can be located by drawing an horizontal line at the desired value of \( J/M^2 \), as explained in the caption.)

Stars spinning above a given threshold are not stable against mass shedding [63]. Instability arises when
the centrifugal force is strong enough to disrupt the star. In Newtonian gravity, the equatorial mass shedding frequency is approximately the Keplerian frequency \( \Omega_K = (M/R)^{3/2} \). Although corrections to the Keplerian frequency are expected in a general relativistic framework, \( \Omega_K \) provides a good estimator for the validity of

\[ ds^2 = -f dt^2 + \frac{k}{f} \left[ g (dr^2 + r^2 d\theta^2) + r^2 \sin^2 \theta (d\varphi - \zeta(r) dt)^2 \right] \]

where \( f = \frac{1}{2} \frac{\sigma^2}{\sqrt{\Lambda}} \), \( \Lambda = \frac{\mu^2}{\sqrt{\Lambda}} \), and \( \zeta(r) \) is a function of \( r \).

the slow-rotation approximation. (See Ref. 66 for a comparison of the slow-rotation regime vs. full numerical results.) In the following, the slow-rotation approximation will be considered valid for \( \Omega/\Omega_K < 1 \). Numerical results extend up to \( \Omega \sim \Omega_K \).

**B. Boson stars**

A well-known example of nonrotating boson star is the model by Colpi, Shapiro and Wasserman (CSW) 23. This is based on the Lagrangian density

\[ \mathcal{L}_{CSW} = -\frac{1}{2} g^{\mu\nu} \Phi^{*}_{\mu} \Phi_{\nu} - \frac{1}{2} m_B^2 |\Phi|^2 - \frac{1}{4} \lambda |\Phi|^4, \quad (2.28) \]

where \( \Phi \) is a complex self-interacting scalar field, \( m_B \) is the boson mass and \( \lambda \) is a dimensionless coupling constant determining the strength of the self-interaction. Since we are interested on star masses of the order of the Chandrasekhar mass, we assume \( \lambda \gg 1 \). Setting

\[ \Phi(t, r) = \frac{M_{\text{planck}}}{\sqrt{4\pi}} \sigma(r)e^{-i\omega t}, \quad (2.29) \]

the structure of the boson star is determined by

\[ \left( x_0 - x_s \right) t = \frac{x_0^2}{2} \left( 3 \frac{\Omega^2}{B} + 1 \right) \left( \frac{\Omega^2}{B} - 1 \right)^2, \quad (2.30) \]

\[ \frac{B'}{f B x_s} - \frac{1}{x_s^2} \left( 1 - \frac{1}{f} \right) = \frac{1}{2} \left( \frac{\Omega^2}{B} - 1 \right)^2, \quad (2.31) \]

\[ \sigma = \frac{1}{\sqrt{\Lambda}} \left( \frac{\Omega^2}{B} - 1 \right)^{1/2}. \quad (2.32) \]

where \( x_s = m_B r/\sqrt{\Lambda}, \quad \Lambda = \frac{\mu^2}{\sqrt{\Lambda}} \), and \( \mu \) is the boson mass. The method to build rotating solutions of Sect. 3 yields inconsistencies because the angular momentum of the boson star is quantized, thus slowly rotating solutions are forbidden [67, 68]. A variation of the CSW model which allows for rotating solutions is the Kleihaus, Kunz, List and Schaffer (KKLS) model [69]. The KKLS solution is based on the self-interacting complex scalar field with Lagrangian density

\[ \mathcal{L}_{KKLS} = -\frac{1}{2} g^{\mu\nu} \left( \Phi^{*}_{\mu,\nu} \Phi_{\nu} + \Phi^{*}_{\nu,\mu} \Phi_{\mu} \right) - U(|\Phi|), \quad (2.33) \]

where \( U(|\Phi|) = \lambda |\Phi|^2(|\Phi|^4 - a|\Phi|^2 + b) \). The mass of the boson is given by \( m_B = \sqrt{\lambda} b \). The equations for the boson star structure can be solved by setting

\[ ds^2 = -f dt^2 + \frac{k}{f} \left[ g (dr^2 + r^2 d\theta^2) + r^2 \sin^2 \theta (d\varphi - \zeta(r) dt)^2 \right] \quad (2.34) \]
and $\Phi = \phi e^{i\omega t + in\varphi}$, where the metric components and the real function $\phi$ depend only on $r$ and $\theta$. The requirement that $\Phi$ is single-valued implies $n = 0, \pm 1, \pm 2, \ldots$. The solution has spherical symmetry for $n = 0$ and axial symmetry otherwise. The mass $M$ and the angular momentum $J$ can be read off from the asymptotic expansion of $f$ and $\zeta$,

$$M = \frac{1}{2G} \lim_{r \to \infty} r^2 \partial_r f, \quad J = \frac{1}{2G} \lim_{r \to \infty} r^3 \zeta,$$

respectively. Since the Lagrangian density is invariant under a global $U(1)$ transformation, the current $j^\mu = -i\Phi^* \partial^\mu \Phi + \text{c.c.}$ is conserved. The associated charge is

$$Q = 4\pi \omega_s \int_0^\infty \int_0^\pi |g|^{1/2} \frac{1}{f} \left(1 + \frac{n \omega_s}{\omega_s r}\right) \phi^2 dr d\theta.$$

The angular momentum $J$ and the scalar charge $Q$ are related by $J = nQ$. The numerical procedure to extract the metric and the scalar field is described in Ref. [63]. Throughout the paper we will consider solutions with $n = 2$, $b = 1.1$, $\lambda = 1.0$, $a = 2.0$ and different values of $(J, M) = (3781, 1296)$, $(3400, 1081)$, $(2800, 906)$, corresponding to $J/(GM^2) \sim 0.566$, $0.731$ and $0.858$, respectively. The $n = 1$ solutions in Ref. [63] exhibit similar features. The two top panels of Fig. 3 show the metric functions for boson stars with $J/(GM^2) \sim 0.566$ and $0.858$ along the equatorial plane. The change in the metric potentials from $\theta = \pi/2$ to $\theta = \pi/4$ for these solutions is plotted in the bottom panels of Fig. 3. The metric functions do not depend significantly on the longitudinal angle. Figure 4 gives $g_{tt}$ as a function of distance for the case with $J/(GM^2) \sim 0.566$ at the equator. The behavior of $g_{tt}$ demonstrates that boson stars develop ergoregions deeply inside the star. For this particular choice of parameters, the ergoregion extends from $r/(GM) \sim 0.0471$ to $0.770$. A more complete discussion on the ergoregions of rotating boson stars can be found in Ref. [63].

### C. Superspinars and Kerr-like wormholes

Following Ref. [33], a superspinar can be modeled as a Kerr spacetime

$$ds_{\text{Kerr}}^2 = - \left(1 - \frac{2Mr}{\Sigma}\right) dt^2 + \frac{\Sigma}{\Delta} dr^2 + \left(r^2 + a^2 \sin^2 \theta + \frac{2Mr}{\Sigma} a^2 \sin^4 \theta\right) d\phi^2 + \left(\frac{r^2 + a^2 \sin^2 \theta}{\Sigma} - \frac{4Mr}{\Sigma} a \sin^2 \theta d\phi dt + \Sigma d\theta^2, \tag{2.37}\right.$$n

where

$$\Sigma = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 + a^2 - 2Mr. \tag{2.38}\right.$$n

Unlike Kerr black holes, superspinars have $a > M$ and no horizon. Since the domain of interest is $-\infty < r < +\infty$, the spacetime possesses naked singularities and closed timelike curves in regions where $g_{\phi\phi} < 0$. Higher-energy modifications in the vicinity of the singularity are expected. Thus, following Ref. [33] a small region around the origin is assumed to be excised or modified by stringy corrections. The most popular excision method uses domain walls formed by supertubes [33, 70]. No explicit solution describing a superspinar in four-dimensional spacetime is known.

Kerr-like wormholes are described by metrics of the form

$$ds_{\text{wormhole}}^2 = ds_{\text{Kerr}}^2 + \epsilon_{ij} dx^i dx^j, \tag{2.39}\right.$$n

where $\epsilon_{ij}$ is infinitesimal. In general, Eq. (2.39) describes an horizonless object with excision at some small distance of order $\epsilon$ from the would-be horizon. (See Ref. [31] for details on nonrotating wormholes.) Wormholes require exotic matter and/or divergent stress tensors. Usually, some ultra-stiff matter is assumed close to the would-be horizon.

Both superspinars and rotating wormholes are described by Kerr metrics. Superspinars can be modeled with a rigid “wall” at finite radius $r_0$, which excludes the pathological region. Wormholes and other Kerr-like objects can be modeled as “star bombs” [53, 55] with walls at fixed Boyer-Lindquist radius $r_0 = r_+(1 + \epsilon)$.

### III. Ergoregion Instability for Rotating Stars

The stability of BH doubles can be studied perturbatively by considering small deviations around equilibrium. As explained in the introduction, we consider only scalar perturbations. This is justified as follows. Axial gravitational perturbations are described in the large $l = m$ limit by the same equation of scalar perturbations. In this regime, our results describe both kinds of perturbations. In the small $l = m$ limit, gravitational perturbations are expected to be more unstable than scalar perturbations because gravitational perturbations interact more strongly with the ergoregion. For black holes, the superradiant amplification of spin-2 fields is much stronger than the superradiant amplification of other fields. This conclusion is also verified under certain simplifying assumptions in Ref. [71].) Thus, scalar perturbations provide a lower bound on the strength of the instability.

#### A. Axial gravitational perturbations for perfect fluid stars

In the large $l = m$ regime, axial gravitational perturbations [71, 72] are described by a simple equation. The
FIG. 3: Top panels: Metric coefficients for rotating boson stars along the equatorial plane. The parameters are $n = 2$, $b = 1.1$, $\lambda = 1.0$, $a = 2.0$, $J/(GM^2) \sim 0.566$ (left panel) and $J/(GM^2) \sim 0.858$ (right panel). Bottom panels: Fractional difference of the metric potentials between $\theta = \pi/2$ and $\theta = \pi/4$ for rotating boson stars with angular momentum $J/(GM^2) \sim 0.566$ (left panel) and $J/(GM^2) \sim 0.858$ (right panel). The plots give the maximum possible fractional difference between these quantities.

The full metric is a perturbation of Eq. (2.19) [73]:

$$ds^2 = ds_0^2 + 2 \sum_{lm} \left( h_{0lm}^m(t,r) + h_{1lm}^m(t,r) \right) \times \left( -\sin^{-1} \theta \partial_\phi Y_{lm} d\theta + \sin \theta \partial_\theta Y_{lm} d\phi \right), \quad (3.1)$$

where $ds_0^2$ is the unperturbed metric (2.19) and $Y_{lm}$ are scalar spherical harmonics. The quantities

$$h_{0lm}^m = \sqrt{f(r)/B(r)}K_6, \quad h_{1lm}^m = \sqrt{B(r)/f(r)}V_4,$$

satisfy the system of equations (see Eqs. (13)-(16) in Ref [72])

$$K_3' = 16\pi(p + \rho)u_3 - \frac{2K_3}{r} + \frac{l^2 + l - 2}{r^2}K_6 - \frac{2m\zeta'V_4}{(l^2 + 1)f}, \quad (3.3)$$

$$K_6' = -\frac{B}{f}(-\omega + m\zeta)V_4 - \left( \frac{f'}{2f} - \frac{B'}{2B} - \frac{2}{r} \right)K_6 + BK_3, \quad (3.4)$$

where $K_3$ and $K_6$ are two extrinsic curvature variables and

$$V_4 = \frac{r^2}{l^2 + 2} \left( (-\omega + m\zeta)K_3 - \frac{2m\zeta'}{l(l+1)}B \right), \quad (3.5)$$

$$u_3 = \frac{2m(\Omega - \zeta)}{2m(\Omega - \zeta) - l(l+1)(-\omega + m\Omega)}K_6. \quad (3.6)$$

In the large $l = m$ limit, Eqs. (3.3) and (3.4) reduce to

$$K_3' = -\frac{2}{r}K_3 + \frac{m^2}{r^2}K_6, \quad (3.7)$$

$$K_6' = BK_3 - \frac{B}{f}(\Sigma + \zeta)^2r^2K_3 - \left( \frac{f'}{2f} - \frac{B'}{2B} - \frac{2}{r} \right)K_6, \quad (3.8)$$
FIG. 4: The $g_{tt}$ metric coefficient for a boson star with $J/(GM^2) \sim 0.566$ at its equator. The ergoregion is identified by the region inside the dotted vertical lines and extends from $r/(GM) \sim 0.047$ to 0.77.

where $\Sigma = -\omega/m$. Combining Eqs. (3.7)-(3.8) and neglecting terms of order $1/m^2$, it follows

$$K''_3 + m^2 B \left( (\Sigma + \zeta)^2 - \frac{f}{r^2} \right) K_3 = 0.$$  \hspace{1cm} (3.9)

This equation also describes the scalar perturbations of gravastars, as it will be shown below.

B. Scalar field instability for slowly rotating gravastars: WKB approach

Consider now a minimally coupled scalar in the background of a gravastar. The metric of gravastars is given by Eq. (2.19). In the large $\ell = m$ limit, which is appropriate for a WKB analysis [58, 74], the scalar field can be decomposed as

$$\Phi = \bar{\chi}(r)e^{-\frac{i}{2} \int \left( \frac{\Sigma}{f} + \frac{\Sigma}{f_{\Sigma}} \right) dr} e^{-i\omega t} Y_{lm}(\theta, \phi).$$  \hspace{1cm} (3.10)

The function $\bar{\chi}$ is determined by the Klein-Gordon equation, which yields

$$\bar{\chi}'' + m^2 T(r, \Sigma) \bar{\chi} = 0,$$  \hspace{1cm} (3.11)

where $\Sigma$ is defined as below Eq. (3.8) and

$$T = \frac{B(r)}{f(r)} (\Sigma - V_+) (\Sigma - V_-),$$  \hspace{1cm} (3.12)

$$V_{\pm} = -\zeta \pm \sqrt{f(r)/r}.$$  \hspace{1cm} (3.13)

The derivation of Eq. (3.11) uses Eq. (3.10) and terms of order $O(1/m^2)$ are dropped. Equation (3.11) can be shown to be identical to Eq. (3.5) describing axial gravitational perturbations of perfect fluid stars. Thus the following results apply to both kinds of perturbations.

The eigenfrequencies of Eq. (3.11) can be computed in the WKB approach following Ref. [58]. This method is in excellent agreement with full numerical results [54, 74]. The quasi-bound unstable modes are determined by

$$m \int_{r_n}^{r_b} \sqrt{T(r)} dr = \frac{\pi}{2} + n\pi, \hspace{0.5cm} n = 0, 1, 2, \ldots$$  \hspace{1cm} (3.14)

FIG. 5: Top panel: Potentials $V_{\pm}$ for the thin-shell gravastar with $r_2 = 1.3$, $r_1 = 1$, $w_1 = 50$ and $w_2 = 1$. The ergoregion extends from $r \sim 0.247$ to 0.832 and corresponds to a gravastar with $J \sim 0.333M^2$ and $M \Omega \sim 0.105$. Bottom panel: Potentials for the anisotropic pressure gravastar with $r_2 = 2.2$, $r_1 = 1.8$ and $M = 1$. The ergoregion extends from $r \sim 0.270$ to $r \sim 1.055r_2$ and rotates with angular momentum $J/M^2 = 1.00$, corresponding to $\Omega \sim 0.250$. The quasi-bound unstable modes are determined by
and have timescale
\[ \tau = 4e^{2m} \int_{r_a}^{r_b} \sqrt{T} dr \int_{r_a}^{r_b} \frac{d}{d\Sigma} \sqrt{T} dr , \] (3.15)

where \( r_a, r_b \) are solutions of \( V_+ = \Sigma \) and \( r_c \) is determined by the condition \( V_- = \Sigma \).

The potentials \( V_+ \) are displayed in Fig. 5 for the gravastars models of Sect. II A. The top panel shows the potential for the thin-shell model with \( r_2 = 1.3, r_1 = 1, w_1 = 50 \) and \( w_2 = 1 \). The gravastar rotates with angular frequency \( \Omega \sim 0.105 \) and the ergoregion lies in the region \( r \sim (0.247, 0.832)r_2 \). The bottom panel refers to the anisotropic pressure model with \( r_2 = 2.2, r_1 = 1.8 \) and \( M = 1 \). The ergoregion extends from \( r \sim 0.270r_2 \) to \( r \sim 1.055r_2 \) and rotates with angular frequency \( \Omega \sim 0.250 \).

![FIG. 6: Details of the ergoregion instability (\( m = 1 \) and \( m = 4 \)) for the thin-shell gravastar of Sect. II A with \( r_2 = 1.3, r_1 = 1, w_1 = 50 \) and \( w_2 = 1 \). The plot shows the logarithm of the dimensionless instability timescale \( \tau/M \), the dimensionless angular velocity \( M\Omega \) and the oscillation frequency \( \omega \) vs. the angular momentum per unit mass, \( J/M^2 \).](image)

The results of the WKB computation are shown in Fig. 6 and Tables I-III. Figure 6 displays the results for the least compact thin-shell gravastar of Sect. II A with \( m = 1, 4 \). Although the WKB approximation breaks down at low \( m \) values, these results should still provide reliable estimates [58]. This statement will be verified in Sect. III B 1 with a full numerical integration of the Klein-Gordon equation. Table II compares three different gravastars for \( J/M^2 = 1 \) and \( m = 1, 2, \ldots, 5 \). The results show that the instability timescale increases as the star becomes more compact. Table III refers to the most compact thin-shell gravastar for various angular frequencies. The instability timescale depends strongly on the rotation. A good fit for the instability timescale is

\[ \log \tau \sim a + b\sqrt{J/M^2} + cJ/M^2 , \] (3.16)

where \( a = 68.0, b = -76.7, c = 26.2 \) for \( m = 5 \) and \( a = 55.8, b = -61.4, c = 21.0 \) for \( m = 4 \). Table III shows the WKB results for the anisotropic pressure model for different values of \( J/M^2 \). Larger values of \( J/M^2 \) make the star more unstable. The instability timescales are fitted by

\[ \log \tau \sim a + b(J/M^2)^c . \] (3.17)

where \( a = -21.9, b = 39.2, c = -0.39 \) for \( m = 5 \) and \( a = -13.7, b = 28.5, c = -0.43 \) for \( m = 4 \).

Both models have similar low-\( m \) behaviors. It will be shown in Sect. III B 1 that the WKB results for the instability timescale differ from the numerical results by about one order of magnitude at low \( m \). On the contrary, the resonant frequencies match well the WKB results even for low-\( m \) modes. Calculations show that the resonant frequency is \( \text{Re}(\omega) \sim \alpha\Omega \), where \( \alpha \sim 1.1 - 1.2 \).

The maximum growth rate of the instability is of the order of a few thousand \( M \), at least for large \( J \). This instability is crucial for the star evolution. A comparison of Table IV with Tables 1 and 2 in Ref. [58] shows that the ergoregion instability of gravastars is stronger than the ergoregion instability of uniform density stars by many orders of magnitude. This seems to be a general feature of all BH doubles. Gravitational perturbations are expected to be even more unstable. This will be verified below for superspinars, for which gravitational instability timescales are typically 4–5 orders of magnitude smaller than scalar instability timescales. This is in agreement with the estimates of Ref. [51].

1. Comparison with numerical results

An accurate computation of instabilities requires a numerical solution of the Klein-Gordon equation. However, the WKB approximation provides reliable estimates of the numerical results. The exact potential \( T \) of Eq. (3.13)
\[ \bar{\mathcal{N}} = -\frac{(l+1)B}{r^2} + \frac{B(\omega - m\zeta)^2}{r^2} + \frac{B''}{2rB} + \frac{B'''}{4B} - \frac{5B'^2}{16B^2} - \frac{f'}{2rf} + \frac{B'f'}{8Bf} + 3f'' - \frac{f'''}{4f} \quad (3.18) \]

The results of the numerical integration for the anisotropic pressure gravastar with \( r_2 = 2.2, r_1 = 1.8 \) and \( J/M^2 = 1 \) are shown in Table IV.

<table>
<thead>
<tr>
<th>( J/M^2 )</th>
<th>( J/M^2 )</th>
<th>( J/M^2 )</th>
<th>( J/M^2 )</th>
<th>( J/M^2 )</th>
<th>( J/M^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.40</td>
<td>0.50</td>
<td>0.60</td>
<td>0.70</td>
<td>0.80</td>
<td>0.90</td>
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<td>( J/M^2 )</td>
<td>( J/M^2 )</td>
<td>( J/M^2 )</td>
</tr>
<tr>
<td>0.28</td>
<td>0.35</td>
<td>0.42</td>
<td>0.50</td>
<td>0.57</td>
<td>0.64</td>
</tr>
<tr>
<td>1</td>
<td>10.628 \times 10^4</td>
<td>13.370 \times 10^4</td>
<td>16.238 \times 10^4</td>
<td>18.164 \times 10^4</td>
<td>20.191 \times 10^4</td>
</tr>
<tr>
<td>2</td>
<td>11.324 \times 10^4</td>
<td>14.115 \times 10^4</td>
<td>16.832 \times 10^4</td>
<td>18.321 \times 10^4</td>
<td>21.124 \times 10^4</td>
</tr>
<tr>
<td>3</td>
<td>11.688 \times 10^4</td>
<td>14.399 \times 10^5</td>
<td>16.966 \times 10^5</td>
<td>18.363 \times 10^5</td>
<td>21.130 \times 10^5</td>
</tr>
<tr>
<td>4</td>
<td>11.875 \times 10^5</td>
<td>14.134 \times 10^6</td>
<td>16.203 \times 10^6</td>
<td>18.469 \times 10^6</td>
<td>21.131 \times 10^6</td>
</tr>
<tr>
<td>5</td>
<td>11.458 \times 10^11</td>
<td>13.594 \times 10^10</td>
<td>16.412 \times 10^10</td>
<td>18.676 \times 10^10</td>
<td>21.145 \times 10^10</td>
</tr>
</tbody>
</table>

The WKB approximation for the real part of the frequency shows a remarkably good agreement with the numerical results even at low values of \( m \). For any \( m > 1 \) this agreement is better than 5%. The instability timescale seems to be more sensitive to the details of the WKB integration, with a level of agreement similar to that reported in Ref. [59]. The above results show that the WKB approximation correctly estimates the instability timescale for all values of \( m \) within an order of magnitude.

C. Scalar field instability for rotating boson stars: WKB approach

Consider a scalar field \( \Psi \) minimally coupled to the rotating boson star geometry (not to be confused with the scalar field which makes up the star). Since the metric coefficients depend on \( r \) and \( \theta \), the Klein-Gordon equation cannot be reduced, in general, to a one-dimensional problem. Separation of variables can be achieved by requiring \( g = 1, f = f(r, \pi/2), k = k(r, \pi/2) \) and \( \zeta = \zeta(r, \pi/2) \). These assumptions are justified as follows. Firstly, the metric function \( g \) is very close to unity throughout the entire coordinate region, as can be seen in Figure 3. Since the Klein-Gordon equation does not depend on the derivatives of \( g \), it seems safe to set \( g = 1 \) in the whole domain. Secondly, the angular dependence of the metric coefficients is negligible for slow rotations. The largest variation for one revolution around the star is that of \( f \), which is less than 100% for most cases (see Fig. 5). Moreover, this dependence is extremely weak for most of the values of \( r \). Thirdly, perturbations are localized around the equator in the large \( l = m \) behavior. Therefore, evaluating the metric coefficients at the equator provides a good approximation to the problem. The results obtained in this section are expected to give an order of magnitude estimate for the instability.

The equation for the scalar field is obtained by expand-
TABLE IV: Comparison between analytical and numerical results for anisotropic pressure gravastars with $J/M^2 = 1$, $r_2 = 2.2$, $r_1 = 1.8$ and $M = 1$. The numerical results for the real part are in good agreement with the WKB results. The agreement is better for larger values of $m$. The imaginary parts agree within an order of magnitude.

| $m$ | Analytical (A) Numerical (N) | $\frac{|\omega^N - \omega^A|}{\omega^A}$ |
|-----|-----------------------------|---------------------------------|
| 1   | 15.0, 2.34 x 10^3 21.9, 2.47 x 10^3 31.5% |
| 2   | 18.0, 2.34 x 10^4 18.6, 1.81 x 10^3 3.2% |
| 3   | 19.0, 2.49 x 10^5 18.7, 2.53 x 10^6 1.6% |
| 4   | 19.6, 2.74 x 10^6 19.0, 3.33 x 10^7 3.2% |
| 5   | 19.8, 3.07 x 10^7 19.3, 3.76 x 10^8 2.6% |

FIG. 7: Potentials $V_\pm$ for the boson star model with $J/(GM^2) = 0.566$. The ergoregion extends from $r/(GM) \sim 0.0478$ to 0.779.

and the latter as

$$\Psi = \bar{\Psi}(r)e^{-\frac{i}{2}f \left( \frac{x}{\sqrt{\lambda}} \right)} e^{-i\omega t} Y_{lm}(\theta, \phi).$$  \hspace{1cm} (3.19)

In the large $l = m$ limit, $\bar{\Psi}$ is determined by

$$\bar{\Psi}'' + m^2 T(r, \Sigma) \bar{\Psi} = 0,$$  \hspace{1cm} (3.20)

where

$$T = \frac{k}{f^2} (\Sigma - V_+) (\Sigma - V_-),$$  \hspace{1cm} (3.21)

$$V_\pm = \zeta \pm \frac{f}{r \sqrt{\lambda}}, \quad \Sigma \equiv \frac{\omega}{m},$$  \hspace{1cm} (3.22)

and terms of order $O(1/m^2)$ have been neglected. The potentials $V_\pm$ are plotted in Fig. 7 for the rotating boson star with $J/(GM^2) = 0.566$. The results of the WKB computation are summarized in Table V for $m = 1, 2, \ldots, 5$. The maximum growth rate for the ergoregion instability is of order of $10^3 M$.

TABLE V: Instability for rotating boson stars with parameters $n = 2$, $b = 1.1$, $\lambda = 1.0$, $a = 2.0$ and different values of $J$: $J/(GM^2) = 0.566, 0.731$ and 0.858. The Newton constant is defined as $4\pi G = 0.05$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\omega_{\text{MWB}}$</th>
<th>$\omega_{\text{Q}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>31.8, 8.47 x 10^7 66.6, 3.03 x 10^8</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>26.7, 5.07 x 10^8 13.5, 5.82 x 10^7 6.81, 1.47 x 10^6</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>26.7, 6.27 x 10^8 16.9, 9.27 x 10^8 3.4, 2.81 x 10^9</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>26.7, 5.82 x 10^8 17.1, 60 x 10^8 4.9, 2.81 x 10^9</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>26.7, 5.55 x 10^8 18.2, 915 x 10^8 5.7, 1.17 x 10^12</td>
<td></td>
</tr>
</tbody>
</table>

IV. INSTABILITY OF WORMHOLES AND OTHER KERR-LIKE OBJECTS

The instability of wormholes and other Kerr-like objects is studied by considering Kerr geometries with arbitrary rotation parameter $a$ and a “mirror” at some Boyer-Lindquist radius $r_0$.

Using the Killersley tetrad and Boyer-Lindquist coordinates, the radial and the angular equations for a spin-$s$ field in the Kerr metric are, respectively, [82]

$$\Delta^{-s} d^r \left( \Delta^{s+1} dR_{lm} \right) + \left( K^2 - 2is(r - M)K \Delta + 4is\omega r - \lambda \right) R_{lm} = 0,$$  \hspace{1cm} (4.1)

$$\left[ (1 - x^2) s S_{lm, x} + \left( (cx)^2 - 2c^2 x + s^2 + s A_{lm} - \frac{(m + sx)^2}{1 - x^2} \right) S_{lm} = 0 \right],$$  \hspace{1cm} (4.2)

where $x = \cos \theta$, $\Delta = r^2 - 2Mr + a^2$, $K = (r^2 + a^2)\omega - am$ and the separation constants $\lambda$ and $s A_{lm}$ are related by

$$\lambda = s A_{lm} + a^2 \omega^2 - 2am \omega.$$  \hspace{1cm} (4.3)

The spacetime possesses two horizons located at $r_+ = M \pm \sqrt{M^2 - a^2}$. The equations above can be solved analytically in two regimes: Small $\omega M$ [52, 55, 86] and near-extremal regime [52], $r_+ \sim r_-$ and $\omega \sim m \Omega_h$, where $\Omega_h \equiv a/(2M_{+})$ is the angular velocity at the horizon.

A. Slowly rotating star bombs

The strategy for slowly rotating objects is to approximate the Teukolsky equation close to the horizon and then match it to an approximate solution near infinity. This procedure is made possible by an overlap between the regions of validity of each approximation. Details of the computation are in Appendix A.

In the slow-rotation approximation, the spheroidal wavefunctions (4.2) reduce to the spin-weighted spherical harmonics with eigenvalues $s A_{lm} = l(l + 1) - s(s + 1)$. Matching the inner to the outer solution, the condition for the frequency $\omega$ is (see Appendix A for details)
Equation (4.7) can be solved in the limit \( a(l) \) respectively. The mode scheme, i.e., \( m \) and \( \omega \), respectively. The solutions are consistent with the approximation characteristic values of Equation (4.4) can be solved numerically for the characteristic values of \( l \approx 1, \alpha \ll M \) and \( z_0 \approx 0 \). The instability timescale for scalar perturbations is \( \tau / M = 2 \left( 1 - \frac{1}{2} \frac{a}{M}^2 + \sqrt{1 - \frac{a}{M}^2} \right) \times \sqrt{1 - \left( \frac{a}{M} \right)^2} \times \log^{-1} \left( \frac{1 + \gamma (\omega_n - m \Omega)}{1 - \gamma (\omega_n - m \Omega)} \right) \). (4.6)

In Table VI we show the results for \( s = 0 \) for different values of \( l = m \) values of the scalar field, respectively. The mode \( l = m = 1 \) is absent.

<table>
<thead>
<tr>
<th>( l = m )</th>
<th>( \Re(\omega) / M )</th>
<th>( \tau / M )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.02 ( \times 10^{13} )</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.09 ( \times 10^{13} )</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.17 ( \times 10^{13} )</td>
<td></td>
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</tbody>
</table>

The relation between the position of the mirror and the frequency of the wave is (see Appendix B for details)

\[
(-z_0)^{-s-i\kappa} = (x_0/\sigma)^{-s-i\kappa} = \frac{R_1 + pR_3(-2i\omega\sigma)^{2i\delta}}{R_2 + pR_4(-2i\omega\sigma)^{2i\delta}}.
\]

Numerical solutions of the above equation for an object with \( a = 0.998M \) are shown in Fig. 8. The real part of the characteristic frequencies is always close to \( m \Omega \) and \( \Im(\omega) \ll \Re(\omega) \). The results are thus consistent with the initial assumptions. The instability timescale for gravitational perturbations is about four orders of magnitude smaller than scalar perturbations.

C. Numerical results

The oscillation frequencies of the modes can be found from Eq. (4.11) written in the canonical form

\[
\frac{d^2Y}{dr_*^2} + VY = 0,
\]

where

\[
Y = \Delta s/2 (r^2 + a^2)^{1/2} R,
\]

\[
V = \frac{K^2 - 2is(r - M)K + \Delta(4ir\omega s - \lambda)}{(r^2 + a^2)^2} - G^2 - \frac{dG}{dr_*},
\]

and \( K = (r^2 + a^2)\omega - am, G = s(r - M)/(r^2 + a^2) + r\Delta(r^2 + a^2)^{-2} \). The separation constant \( \lambda \) is related to the eigenvalues of the angular equation by Eq. (4.13). The eigenvalues \( sA_{lm} \) are expanded in \( a \omega \) as

\[
sA_{lm} = \sum_{k=0}^{\infty} f_{k}^{(k)}(a \omega)^{k},
\]

where terms up to order \( (a \omega)^2 \) are included. Demanding only outgoing waves at infinity implies:

\[
Y \sim r^{-s} e^{i\omega r}.
\]

Numerical results are obtained by integrating Eq. (4.11) inward from a large distance \( r_\infty \). The integration is performed with the Runge-Kutta method with fixed \( \omega \) starting at \( M_\infty = 400 \), where we impose the asymptotic behavior (4.13). (Choosing a different initial point does not affect the final results.) The numerical integration is stopped at the radius of the mirror \( r_0 \), where the value of the field \( Y(\omega, r_0) \) is extracted. The integration is repeated for different values of \( \omega \) until \( Y(\omega, r_0) = 0 \) is found.
to a certain precision. If $Y(\omega, r_0)$ vanishes, the field satisfies the boundary condition for perfect reflection; $\omega = \omega_0$ is the oscillation frequency of the mode.

Typical results for scalar perturbations are summarized in Table VII and Figs. 9–11. The top panels of Fig. 9 show the imaginary and real parts of the fundamental mode frequency for $a = 0.998M$ vs. the mirror position $r_0 = r_+ (1 + \epsilon)$ for different $l = m$ values, respectively. The instability is weaker for larger $m$ (even for $l \neq m$, although this is not shown in the plots). Comparisons with the analytical results in the near-extremal regime are shown in the bottom panels. The numerical integrations show that $\Re(\omega) \sim m\Omega$ in agreement with the analytical results of Sect. IV B. The instability timescales are consistent with the analytical results within a factor $\sim 3$. The minimum instability timescale is of order $\tau \sim 10^5 M$ for a wide range of mirror locations.

Figure 10 shows the results for gravitational perturbations. Instability timescales are of the order of $\tau \sim 2 - 6M$. Thus gravitational perturbations lead to an instability about five orders of magnitude stronger than the scalar perturbations (see Table VII). Figure 11 shows that the ergoregion instability remains relevant even for values of the angular momentum as low as $a = 0.6M$.

Some features of these results are intriguing and deserve further study. For instance, instabilities are found for $\Re(\omega) > m\Omega$, although superradiant instability should be confined to the superradiant regime. A possible explanation is that other instabilities are present. Numerical results also show that there is a mirror location which maximizes the instability at fixed $a$. It would be interesting to explain the physical meaning of this location.

\begin{table}[h]
\centering
\caption{Characteristic frequencies and instability timescales for a “star bomb” with $a = 0.998M$. The mirror is located at $\epsilon = 0.1$. Results for scalar and gravitational perturbations are shown for different $l = m$ values.}
\begin{tabular}{|c|c|c|}
\hline
$l = m$ & $s = 0$ & $s = 2$ \\
\hline
1 & $(0.1120, 0.6244 \times 10^{-5})$ & $-$ \\
2 & $(0.4440, 0.5373 \times 10^{-5})$ & $(0.4342, 0.2900)$ \\
3 & $(0.7902, 0.1928 \times 10^{-5})$ & $(0.7803, 0.2977)$ \\
4 & $(1.1436, 0.5927 \times 10^{-6})$ & $(1.1336, 0.3035)$ \\
\hline
\end{tabular}
\end{table}

\section{V. Instability of Superspinsars}

Several arguments suggest that objects rotating above the Kerr bound are unstable. Firstly, extremal Kerr BHs are marginally stable. Thus the addition of extra rotation should lead to instability. Secondly, fast-spinning objects usually take a pancake-like form \cite{75} and are subject to the Gregory-Laflamme instability \cite{76, 77}. Finally, Kerr-like geometries, like naked singularities, seem to be unstable against a certain class of gravitational perturbations \cite{78, 79, 80} called "algebraically special perturbations" \cite{81}. These perturbations are described by modes with zero Teukolsky-Starobinsky constant \cite{82}. The Teukolsky function $R_{lm}(r)$ has solution

$$R_{lm}(r) = (A + Br + Cr^2 + Dr^3) \times e^{\omega \alpha^2(r_+ - r) + \alpha \Omega \phi} S_{lm}(\theta),$$

(5.1)

where $A, B, C, D$ are constants and $S_{lm}$ are spin weight-2 spheroidal harmonics \cite{83}. These modes satisfy the correct boundary conditions at infinity and are well behaved for any $r > -\infty$. Although superspinsars require particular boundary conditions at the excised region, unstable algebraically special modes are likely to be present. They can be computed with the continued fraction method \cite{84, 85}. Algebraically special modes correspond to a zero of the Teukolsky-Starobinsky constant squared:

$$|C|^2 = \lambda^2 (\lambda + 2)^2 - 8\omega^2 \lambda \left(\alpha^2 (6 + 5\lambda) - 12\alpha^2\right) + 144\omega^2 \left(\alpha^4 \omega^2 + M^2\right),$$

(5.2)

where $\lambda$ is defined in Eq. (4.3) for $s = -2$ and $\alpha^2 = a^2 - ma/\omega$. (Note a typographical error in the corresponding expressions in Ref. \cite{83}.) The technique of Ref. \cite{84} can be used to evaluate the algebraically special modes at fixed $a$. In the range $0 < a < M$, the modes coincide with those of Ref. \cite{85}. Some results for $a > M$ are listed in Table VIII. The typical timescale is of the order of $10^{-6}$ seconds for a one solar mass BH and 1 second for $M = 10^6 M_\odot$.

\section{VI. Detectability by Earth- and Space-Based Gravitational Wave Detectors}

The ergoregion instability may be of interest for gravitational wave astronomy. Contrary to the $r$-mode instability of neutron stars \cite{88, 89}, the ergoregion instability is not limited to solar mass objects. Thus chances of detection are larger because the signal can fall in frequency bands where the detectors are more sensitive.

\subsection{A. Signal-to-noise ratio}

Detectability depends only on the energy released and the detector frequency bandwidth. The sky-averaged
FIG. 8: Top panels: Imaginary and real parts of the characteristic scalar frequencies for an object with \( a = 0.998M \), according to the analytical prediction for rapidly spinning objects. The mirror location is at \( r_0 = (1 + \epsilon)r_+ \). The real parts are almost constant and close to \( m\Omega \), as required by the assumptions used in the approximation. Bottom panels: Same results for gravitational perturbations. The instability is much stronger for \( s = 2 \) fields.

The signal-to-noise ratio (SNR) is

\[
\rho^2 = \frac{2}{5\pi^2 D^2} \int df \frac{1}{f^2 S_h(f)} \frac{dE}{df}, \quad (6.1)
\]

where \( D \) is the distance to the source and \( S_h(f) \) is the noise power spectral density of the detector. Using \( dE = 2\pi f dJ/m \), the SNR for the \( l = m = 2 \) mode is

\[
\rho^2 = \frac{2}{5\pi D^2} \int df \frac{1}{f S_h(f)} \frac{dJ}{df}. \quad (6.2)
\]

Fits to the resonant frequencies yield

\[
\text{Re}[\omega] = 2\pi f \approx \alpha \Omega, \quad (6.3)
\]

where \( \alpha \sim 1.1 - 1.2 \). From Eq. (6.3) it follows \( dJ/df \approx 2\pi I/\alpha \). Assuming the moment of inertia \( I \sim 2\beta M^3 \) to be roughly independent of the angular velocity \( \beta \sim 1 \) (computations show that \( \beta \sim 1 \) to a very good approximation for gravastars and superspinars near the Kerr bound) Eq. (6.2) can be rewritten as

\[
\rho^2 = \frac{8\beta M^3}{5\alpha D^2} \int_{f_{\text{min}}}^{f_{\text{max}}} df \frac{1}{f S_h(f)}. \quad (6.4)
\]

The minimum and maximum frequencies in the above integral are chosen as \( f_{\text{min}} = 0.9 f_{\text{max}} \). This is a conservative estimate based on a simple model for the evolution of the system. SNRs for objects at 20Mpc distance for LIGO/Advanced LIGO and LISA are shown in Table IX for \( \Omega M = 0.2 \). Solar-mass objects are difficult to detect, although LIGO (Advanced LIGO) could be able to detect objects with \( M \gtrsim 30M_\odot \) (\( M \gtrsim 10M_\odot \)). SNRs of several thousands are easily achieved for supermassive objects;
FIG. 9: Top panels: Numerical results for the imaginary part (left panel) and real part (right panel) of the fundamental mode frequency vs. the mirror position $r_0 = r_+ (1 + \epsilon)$ for different $l = m$ values. The angular momentum is $a = 0.998M$. The instability grows monotonically with $l = m$. Bottom panels: Resonant frequencies obtained through numerical integration of the Teukolsky equation (solid lines) and the analytical solutions in the near-extremal regime (dotted lines). The relative difference of the real part of the resonant frequency and $m\Omega$ is plotted in the bottom right panel, showing a remarkable agreement between analytical and numerical results. The analytical results for the imaginary part (bottom left panel) agree with the numerical results within a factor $\sim 3$.

very massive objects could be easily observed by LISA.

B. Waveforms

Templates for matched filtering can be obtained by deriving the time evolution of the instability. The latter proceeds in two steps: A phase characterized by an exponential growth, where the linear approximation is valid, followed by a nonlinear phase. Contrary to the $r$-mode instability, the ergoregion instability does not couple strongly to the fluid composing the object. Therefore, the nonlinear phase is expected to be somewhat different from the $r$-mode saturation phase [58, 59].

As an illustration of waveform estimation, consider instability triggered by a particle in circular orbit around the compact object. Focusing on the $l = 2$ mode, the metric perturbations in the linear perturbation regime have the form

$$h_+ = \frac{M}{D} h_0 e^{t/\tau} \sin (\omega t - 2\phi) \chi_+, \quad (6.5)$$

$$h_\times = \frac{M}{D} h_0 e^{t/\tau} \cos (\omega t - 2\phi) \chi_\times, \quad (6.6)$$
FIG. 10: Imaginary part (left panel) and real part (right panel) of the frequency of some modes vs. the mirror position for a gravitational field with \( l = m = 2, 3, 4 \). The angular momentum of the object is \( a = 0.998M \).

<table>
<thead>
<tr>
<th>( 2M\omega )</th>
<th>( m=0 )</th>
<th>( m=2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>(0, 4.0000)</td>
<td>(0.0000, 4.0000)</td>
</tr>
<tr>
<td>0.10</td>
<td>(0, 4.03107)</td>
<td>(0.918857, 3.66121)</td>
</tr>
<tr>
<td>0.20</td>
<td>(0, 4.13218)</td>
<td>(1.36523, 3.07054)</td>
</tr>
<tr>
<td>0.30</td>
<td>(0, 4.33511)</td>
<td>(1.51483, 2.57005)</td>
</tr>
<tr>
<td>0.40</td>
<td>(0, 4.74699)</td>
<td>(1.53973, 2.19077)</td>
</tr>
<tr>
<td>0.50</td>
<td>(0.715878, 6.57057)</td>
<td>(1.51441, 1.90331)</td>
</tr>
<tr>
<td>0.60</td>
<td>(2.47665, 5.2615)</td>
<td>(1.46928, 1.68066)</td>
</tr>
<tr>
<td>0.70</td>
<td>(2.84578, 4.36141)</td>
<td>(1.41732, 1.50403)</td>
</tr>
<tr>
<td>0.80</td>
<td>(2.93170, 3.71616)</td>
<td>(1.36428, 1.36084)</td>
</tr>
<tr>
<td>0.90</td>
<td>(2.91142, 2.85259)</td>
<td>(1.31270, 1.24256)</td>
</tr>
<tr>
<td>1.00</td>
<td>(2.8632, 2.65299)</td>
<td>(1.26369, 1.14328)</td>
</tr>
<tr>
<td>1.02</td>
<td>(2.80351, 2.78693)</td>
<td>(1.25424, 1.12531)</td>
</tr>
<tr>
<td>1.04</td>
<td>(2.81414, 2.72409)</td>
<td>(1.24490, 1.10790)</td>
</tr>
<tr>
<td>1.10</td>
<td>(2.76243, 2.55079)</td>
<td>(1.21763, 1.05878)</td>
</tr>
<tr>
<td>1.20</td>
<td>(2.67192, 2.30455)</td>
<td>(1.17458, 0.98601)</td>
</tr>
<tr>
<td>1.30</td>
<td>(2.58073, 2.10007)</td>
<td>(1.13444, 0.92268)</td>
</tr>
<tr>
<td>1.40</td>
<td>(2.49185, 1.92767)</td>
<td>(1.09701, 0.86707)</td>
</tr>
</tbody>
</table>

TABLE VIII: Algebraically special modes for various values of the angular momentum.

where \( D \) is the distance to the source, \( h_0 \ll 1 \) and

\[
\chi_+ = \frac{\cos^2 \theta + 1}{2}, \quad \chi_\times = \cos \theta. \quad (6.7)
\]

The above waveforms mimic the Newtonian waveform produced by a small mass in circular orbit around the BH double. The linear perturbation regime corresponds to \( h_{+,\times} < 1 \).

From the discussion in the previous sections, the frequency of the wave is \( \omega \sim \Omega(t) \). The timescale is \( \tau \sim \tau_0(M\Omega)^{-\frac{3}{5}} \), where a dominant \( w \)-mode instability is assumed [71]. The mass can be determined as follows.

The energy carried by the gravitational wave is

\[
\frac{d^2 E}{dt^2 \Omega^2} = \lim_{D \to \infty} \frac{D^2}{16\pi} \left( \dot{h}_+^2 + \dot{h}_\times^2 \right). \quad (6.8)
\]

Assuming that all the energy carried by the gravitational wave is extracted from the star, it follows \( dE/dt = -dM/dt \). The angular momentum radiated in the \( m \)-mode can be obtained from \( dE = \omega dJ/m \). If this angular momentum is also completely extracted from the star, then \( J = I\Omega \), where the moment of inertia \( I = 2\beta M(t)^3 \) is approximately constant in time. Setting \( dJ/dt = m/\omega dE/dt = 2/(\alpha\Omega) dM/dt \), the mass variation rate is

\[
\frac{\dot{M}}{M^2} = \alpha \beta \Omega \dot{\Omega}. \quad (6.9)
\]

Integration of Eq. (6.9) yields

\[
\Omega^2 = \Omega_0^2 + \frac{M_0^{-2} - M^{-2}}{\alpha \beta}. \quad (6.10)
\]

The mass and the angular velocity can be obtained by solving the previous equations with the condition \( dE/dt = -dM/dt \). A typical solution is shown in Figure [12].
FIG. 11: Imaginary part (left panel) and real part (right panel) of the frequency for scalar perturbations (upper panels) and gravitational perturbations (bottom panels).

VII. DISCUSSION

The above results show that BH doubles with high-redshift at their surface are unstable when rapidly spinning. This strengthens the role of BHs as candidates for astrophysical observations of compact highly spinning objects.

Boson stars and gravastars easily develop ergoregion instabilities. Analytical and numerical results indicate that these objects are unstable against scalar field perturbations. Their instability timescale is many orders of magnitude stronger than the instability timescale for ordinary stars with uniform density [58]. Gravitational perturbations are in general about five orders of magnitude stronger than scalar perturbations. This agrees with estimates based on superradiant amplification factors for fields of different spin. In the large $l = m$ approximation, suitable for a WKB treatment, gravitational and scalar perturbations have similar instability timescales.

Since there are no realistic models of superspinars in four dimensions, they have been studied by considering a Kerr geometry above the Kerr bound. This approach shows that superspinars should exhibit various strong instabilities. It would be interesting to repeat the above calculations for a particular realization of superspinars, such as the five-dimensional model mentioned in Ref. [33]. Rotating wormholes which deviate infinitesimally from the Kerr geometry have also been shown to be unstable against scalar and gravitational perturbations.

For $J > 0.4M^2$ instability timescales can be as low as a few tenths of a second for solar mass objects and about a week for supermassive BHs, monotonically decreasing for larger rotations. Therefore, high rotation is an indirect evidence for horizons. The spin of an astrophysical compact object can be estimated by looking at the gas accreting near its surface [2, 91, 92]. A handful of fast-spinning BH-like objects have been reported [6, 7]. The results of this paper suggest that these objects must indeed be BHs.

The ergoregion instability evolution is characterized by very peculiar waveforms. Should a compact astrophysical object evolve through the ergoregion instability, a matched-filtering search could allow gravitational wave
detectors to identify it.

It would be interesting to perform a more detailed analysis of the gravitational sector of the ergoregion instability and derive more refined templates for matched-filtering searches. A first step in this direction would be to compute axial perturbations for any value of the azimuthal number \( m \) along the lines of previous works [7]. Extension of the above results to quark stars or fermion-boson stars [92] is another interesting line of research.

Acknowledgements

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APPENDIX A: ERGOREGION INSTABILITY FOR SLOWLY ROTATING KERR-LIKE OBJECTS (STAR BOMBS)

This appendix contains the analytic computation of the instability of rotating stars bounded by a hard wall.

Massless and massive scalar perturbations and general spin-s perturbations are considered.

1. Massless scalar fields

Following [52, 53, 84], the spacetime outside the object is divided in a near-region, \( r - r_+ \ll 1/\omega \), and a far-region, \( r - r_+ \gg M \). The radial equation (A.1) is solved separately in each of these two regions with the assumptions that the Compton wavelength of the scalar particle is much larger than the typical size of the object, \( 1/\omega \gg M \), and \( a \ll M \). These solutions are then matched in the overlapping region where \( M \ll r - r_+ \ll 1/\omega \) is satisfied. An equation determining the characteristic value \( \omega \) is obtained by imposing suitable boundary conditions at both boundaries. In the following, \( r_+ \) denotes the location of the “would-be” horizon. The location of the mirror or ultra-stiff wall is \( r_0 = r_+(1 + \epsilon) \), \( \epsilon \ll 1 \).

In the far-region, the effects induced by the black hole can be neglected, thus \( a \sim 0, M \sim 0, \Delta \sim r^2 \). The radial wave equation reduces to the wave equation for a massless scalar field of frequency \( \omega \) and angular momentum \( l \) in flat background:

\[
\partial^2_r(rR) + \left[ \omega^2 - l(l+1)/r^2 \right] (rR) = 0. \quad (A1)
\]

The most general solution of this equation is a linear combination of Bessel functions [94]:

\[
R = r^{-1/2} \left[ \alpha J_{l+1/2}(\omega r) + \beta J_{-l-1/2}(\omega r) \right]. \quad (A2)
\]

The behavior of Eq. (A2) for large \( r \) is [94]

\[
R \sim \sqrt{\frac{2}{\pi \omega r}} \left[ \alpha \sin(\omega r - l\pi/2) + \beta \cos(\omega r + l\pi/2) \right], \quad (A3)
\]

while for small \( r \) is [94]

\[
R \sim \alpha \left( \frac{\omega/2}{l+1/2} \right)^{l+1/2} \frac{1}{\Gamma(l+3/2)} r^l + \beta \left( \frac{\omega/2}{l-1/2} \right)^{-l-1/2} \frac{1}{\Gamma(-l+1/2)} r^{-l-1}. \quad (A4)
\]

Demanding only outgoing waves at infinity implies \( \beta = -i\omega e^{i\pi l} \). In the near-region, the radial wave equation is

\[
\Delta \partial_r (\Delta \partial_r R) + r_+^4 (\omega - m\Omega)^2 R - l(l+1)\Delta R = 0. (A5)
\]

Introducing the radial coordinate

\[
z = \frac{r - r_+}{r - r_-}, \quad 0 \leq z \leq 1, \quad (A6)
\]

Eq. (A5) can be rewritten as

\[
z(1-z)\partial_z^2 F + \nonumber + \left[ (1 + i2\omega) - [1 + 2(l+1) + i2\omega] z \right] \partial_z F + \nonumber - \left[ (l+1)^2 + i2\omega(l+1) \right] F = 0, \quad (A7)
\]
where \( R = z^{i_\omega}(1 - z)^{l+1} F \) and the superradiant factor is
\[
\varpi \equiv (\omega - m\Omega)\frac{r_+^2}{r_+ - r_-}.
\] (A8)

Equation (A7) is a standard hypergeometric equation. The most general solution of the near-region equation is
\[
R = A z^{-i_\omega}(1 - z)^{l+1} F(a + c + 1, b + c - 1, z) + B z^{i_\omega}(1 - z)^{l+1} F(a, b, c, z).
\] (A9)

Near the wall, \( r \sim r_+ \), Eq. (A9) is
\[
R \approx A z^{-i_\omega} + B z^{i_\omega}.
\] (A10)

With the usual definition of tortoise coordinate,
\[
r_\ast \equiv \int \frac{r^2 + a^2}{(r - r_+)(r - r_-)} dr,
\] (A11)

Eq. (A10) is cast in the form
\[
R \sim A e^{-i(\omega - m\Omega)\frac{r}{2\pi r_\ast}} + B e^{i(\omega - m\Omega)\frac{r}{2\pi r_\ast}}.
\] (A12)

The large \( r \) behavior of the near-region solution is obtained with the change of variable \( z \to 1 - z \) in the hypergeometric function [94]. The result is:

The matching of the near- and far-region solutions in the region \( M \ll r - r_+ \ll 1/\omega \) yields
\[
\frac{B}{A} = \frac{R_1 + i(-1)^l(\omega(r_+ - r_-))^{2l+1} LR_3}{R_2 + i(-1)^l(\omega(r_+ - r_-))^{2l+1} LR_4},
\] (A14)

where
\[
L = \frac{\pi}{2^{l+2} \Gamma(l + 3/2)\Gamma(2l + 1)\Gamma(l + 1/2)}
\] (A15)

Equation (A14) can be rewritten as
\[
\frac{B}{A} = \prod_{k=1}^{l} \left( \frac{k + 2i\varpi}{k - 2i\varpi} \right) \left( \frac{1 + 2L(r_+ - r_-)^{2l+1} \varpi \left( \prod_{k=1}^{l} (k^2 + 4\varpi^2) \right) \omega^{2l+1}}{1 - 2L(r_+ - r_-)^{2l+1} \varpi \left( \prod_{k=1}^{l} (k^2 + 4\varpi^2) \right) \omega^{2l+1}} \right),
\] (A17)

where it has been used
\[
\frac{R_1}{R_2} = \prod_{k=1}^{l} \left( \frac{k + 2i\varpi}{k - 2i\varpi} \right),
\]
\[
R_3 = iR_1(-1)^{l+1} 2\varpi \prod_{k=1}^{l} (k^2 + 4\varpi^2),
\]
and the analogous relations for \( R_4 \) and \( R_2 \) with \( \varpi \to -\varpi \).

If the mirror is located near the outer horizon at a radius \( r = r_0 \), the scalar field must vanish at the mirror surface. Setting the radial field to zero at \( z = z_0 = z(r_0) \) in the near-region solution, Eq. (A9), it follows
\[
\frac{B}{A} z_0^{2i\varpi} = -\frac{F(l + 1, l + 1 + 2i\varpi, 1 - 2i\varpi, z_0)}{F(l + 1, l + 1 - 2i\varpi, 1 + 2i\varpi, z_0)}.
\] (A18)

The relation between the position of the mirror and the frequency of the scalar wave is obtained from Eq. (A17):
In general, Eq. (A19) must be solved numerically. However, an approximate solution can be obtained by assuming \( \text{Re}(\omega) \gg \text{Im}(\omega) \) and \( \omega \ll 1 \), i.e. a frequency near the superradiant limit \( \omega \approx m\Omega \). This solution gives a good approximation for \( M\omega \ll 1 \), small \( m \) and slowly rotating objects. Since \( \text{Re}(\omega)/\text{Im}(\omega) \ll 1 \), Eq. (A19) can be first solved for real \( \omega \), then a small imaginary part is added and the equation is solved again for \( \text{Im}(\omega) \).

For frequencies near the superradiant limit, the l.h.s. and the last two terms of the r.h.s. of Eq. (A19) are \( \sim 1 \). This yields \( z_0^{2\pi \omega} = 1 \). Using the tortoise coordinate, it follows

\[
e^{2\pi \omega \rho x/(2Mr_+)} = e^{2i\pi \omega x} = 1,
\]

where \( r_+^0 = r_+(z_0) \) and \( x = r_+^0/(2Mr_+) = \log(z_0) \). The solution of Eq. (A20) is

\[
\omega_{n,m} = \frac{\pi(r_+ - r_-)}{r_+^2} n + m\Omega.
\]

Positive frequencies can be obtained by imposing \( x < -n\pi(r_+ - r_-)/(m\Omega r_+^2) \). The superradiant limit requires \( \omega = n\pi(r_+ - r_-)/(r_+^2 x) \ll 1 \). This condition is satisfied by either considering only the fundamental tone and the first overtones, or placing the mirror very close to the horizon, \( |x| \gg 1 \).

By adding a small imaginary part to the resonant frequency, \( \omega = \omega_{n,m} + i\delta \), where \( \delta \ll \omega_{n,m} \), Eq. (A19) becomes

\[
F(l_+ + 1, l + 1 + 2i\omega_0, 1 - 2i\omega_0, z_0) - 2i\omega_0 \omega_{n,m} \omega_{n,m}^{2l+1}
\]

where \( \omega = \omega_0 + i\rho \delta \) and \( \rho = r_+^2/(r_+ - r_-) \). Within the approximations used here, the ratio of the hypergeometric functions in the l.h.s. of Eq. (A22) is \( \sim 1 \). Introducing the tortoise coordinate, it follows

\[
e^{2\rho x \delta} = \frac{1 + 2L(r_+ - r_-)^{2l+1}}{1 - 2L(r_+ - r_-)^{2l+1}} \frac{1 + 2L(r_+ - r_-)^{2l+1} \omega_{n,m} \omega_{n,m}^{2l+1} \left( \prod_{k=1}^{l} (k^2 + 4\omega_0^2) \right)}{1 - 2L(r_+ - r_-)^{2l+1} \omega_{n,m} \omega_{n,m}^{2l+1} \left( \prod_{k=1}^{l} (k^2 + 4\omega_0^2) \right)}
\]

The solution of Eq. (A23) is

\[
\delta = \text{Im}(\omega) = \frac{r_+ - r_-}{2r_+^2 x} \log \left( \frac{1 + \gamma(\omega_{n,m} - m\Omega)}{1 - \gamma(\omega_{n,m} - m\Omega)} \right),
\]

where

\[
\gamma = \frac{\pi}{2} r_+^2 \left( \frac{r_+ - r_-}{2} \right)^2 \prod_{k=1}^{l} (k^2 + 4\omega_0^2) \times
\frac{(\Gamma(l+1))^2}{\Gamma(l+3/2)\Gamma(2l+2)\Gamma(2l+1)\Gamma(l+1/2)\omega_{n,m}^{2l+1}} \geq 0.
\]

Both \( \delta \) and \( \omega_{n,m} \) are very small for \( r_0 \sim r_+ \). However, the argument of the logarithm in Eq. (A24) is \( \sim 1 \). Thus the assumption \( \text{Re}(\omega) \gg \text{Im}(\omega) \) is satisfied.

The above results display two important features. First, Eq. (A24) and Eq. (A25) imply \( \delta \leq 0 \) for \( \text{Re}(\omega) \geq m\Omega \). The time dependence of the scalar field \( \Phi = \exp(-i\omega t) = \exp(-i\text{Re}(\omega) t) \exp(i\delta t) \). Thus the amplitude of the field grows exponentially and the resonant mode becomes instable for \( \text{Re}(\omega) < m\Omega \). Second, \( x < x_{\text{crit}} = -n\pi(r_+ - r_-)/(m\Omega r_+^2) \) and \( \text{Re}(\omega_0) \propto nx^{-1} \) imply \( \delta \leq 0 \) for \( n \leq 0 \). There is always a superradiant amplification for \( n > 0 \) provided that the mirror position is closer than \( x_{\text{crit}} \) to the horizon and the approximations made are satisfied. The critical value \( x_{\text{crit}} \) is positive and outside the domain of \( x \) for \( n < 0 \). In this case the mirror can be located everywhere in the near-region, but there is no superradiant amplification. The growth timescale for \( n > 0 \) is given by

\[
\tau \equiv \delta^{-1} = 2\rho x \log^{-1} \left( \frac{1 + \gamma(\omega_{n,m} - m\Omega)}{1 - \gamma(\omega_{n,m} - m\Omega)} \right),
\]

or, in terms of the physical variables, by Eq. (4.6).
2. Massive scalar field

If the scalar field is massive, the wave equation is

\[
\Delta_\mu \Delta^{\mu} \Psi = \mu^2 \Psi, \quad (A27)
\]

where \( \Delta_\mu \) the covariant derivative and \( \mu \) is the field mass. The above equation is separable. The radial equation is

\[
\Delta \frac{d}{dr} \left( \frac{\Delta dR}{dr} \right) + \left[ \omega^2 (r^2 + \alpha^2) - 4\alpha M \omega r + a^2 \right] \frac{d}{dr} e^{i\omega r} + \left( \alpha^2 - \Delta (\mu^2 - a^2 \omega^2 + \lambda) \right) R = 0. \quad (A28)
\]

Assuming \( a\omega \ll 1, \lambda \approx l(l+1) \) and \( \mu r \ll 1 \), the near-region solution is identical to the massless case. The far-region equation is

\[
d^2 r \left( \left[ \omega^2 - \mu^2 - l(l+1)/r^2 \right] (rR) \right) = 0. \quad (A29)
\]

Defining \( k^2 = \omega^2 - \mu^2 > 0 \), the results of the massless case apply with the substitution \( \omega \rightarrow k \). The matching conditions are

\[
F(l+1, l+1+2i\omega, 1-2i\omega, z_0) \bigg|_{r \rightarrow \infty} \approx z_0^{2l} \prod_{k=1}^{l} \frac{(k+2i\omega)}{(k-2i\omega)} \cdot \frac{1 + 2L(r_+ - r_-)^{2l+1} \omega \left( \prod_{k=1}^{l} (k^2 + 4\omega^2) \right) (\omega^2 - \mu^2)^{l+1/2}}{1 - 2L(r_+ - r_-)^{2l+1} \omega \left( \prod_{k=1}^{l} (k^2 + 4\omega^2) \right) (\omega^2 - \mu^2)^{l+1/2}}.
\]

The large-\( r \) behavior of Eq. (A32) is

\[
R \sim \left( \alpha + \beta \frac{\Gamma(-1-2l)}{\Gamma(1+l-s)} \right) r^{l-s} + \beta \frac{\Gamma(2l+1)}{\Gamma(1+l-s)} r^{-l-s}. \quad (A34)
\]

Demanding only outgoing waves at infinity implies

\[
\beta = -\alpha(-1)^{1+l-s} \frac{\Gamma(2l+2)}{\Gamma(1+l+s)}. \quad (A35)
\]

The near-region behavior of the far-region solution is

\[
R \sim \alpha \left[ \frac{1}{2} r^{l-s} - (2i\omega)^{-1-2l} (-1)^{1+l-s} \times \frac{\Gamma(2l+2)}{\Gamma(1+l-s)} r^{-l-1-s} \right]. \quad (A36)
\]

The radial wave equation in the near-region is

\[
\Delta^{-s} \frac{d}{dr} \left( \Delta^{s+1} \frac{dR}{dr} \right) + \left( \frac{K^2 - 2is(r-M)K}{\Delta} \right) R = 0, \quad (A37)
\]
where \( \lambda = (l - s)(l + s + 1) + O(\omega a) \). Using the approximate relations

\[
\Delta' \approx r_+ - r_-, \quad K = \omega(r^2 + a^2) - am \approx r_+^2(\omega - m\Omega),
\]

\[
K^2 - 2is(r - M)K \approx (r_+ - r_-)^2 \left[ \frac{s^2}{4} + \left( \omega - \frac{s}{2} \right)^2 \right], \tag{A38}
\]

and introducing the radial coordinate \( z \), Eq. (A37) can be written as

\[
z(1-z)\partial_z^2 R + \left[ (s+1) + (s-1)z \right] \partial_z R + \left[ \frac{1-z}{z} \left( \frac{s^2}{4} + (\omega - is/2)^2 \right) + \frac{-s(1-s)(l+s+1)}{1-z} \right] R = 0. \tag{A39}
\]

With the definition \( R = z^s(1-z)^{l+s+1} F \), the previous equation becomes a standard hypergeometric equation \[94\]. Its most general solution is

\[
R = A z^{-s-i\omega} (1-z)^{l+s+1} x \times F(a-c+1, b-c+1, 2-c, z) + B z^s (1-z)^{l+s+1} F(a, c, z), \tag{A40}
\]

where \( a = 1 + l + s + i2\omega \) \( b = l + 1 \) and \( c = 1 + s + i2\omega \). The behavior of Eq. (A40) near \( r_+ \) is

\[
R \sim A e^{-s-i(\omega-m\Omega)} r_+^l \gamma r_+ + B e^{i(\omega-m\Omega)} r_+^l. \tag{A41}
\]

The large-\( r \) behavior is

\[
R \sim \left( \frac{r}{r_+ - r_-} \right)^{l-s} \frac{\Gamma(2l+1)}{\Gamma(l+1)} (A R_1 + B R_2) + \left( \frac{r}{r_+ - r_-} \right)^{-l-s} \frac{\Gamma(-1-2l)}{\Gamma(-l)} (A R_3 + B R_4), \tag{A42}
\]

where

\[
R_1 = -\frac{\Gamma(1-s-2i\omega)}{\Gamma(l+s+1+2i\omega)}, \quad R_2 = \frac{\Gamma(1+s+2i\omega)}{\Gamma(l+s+1+2i\omega)},
\]

\[
R_3 = \frac{\Gamma(1-s-2i\omega)}{\Gamma(-l-s-2i\omega)} = R_1(1)^{-l+1}(s+2i\omega) \prod_{k=1}^l (k^2 + 4\omega^2 - s^2 - 4is\omega),
\]

\[
R_4 = \frac{\Gamma(1+s+2i\omega)}{\Gamma(-l+s+2i\omega)} = -R_2(1)^{-l+1}(s+2i\omega) \prod_{k=1}^l (k^2 + 4\omega^2 - s^2 - 4is\omega). \tag{A43}
\]

The matching of Eq. (A42) and Eq. (A36) yields

\[
\frac{B}{A} = -\prod_{k=1}^l \left( \frac{k+s+2i\omega}{k-s-2i\omega} \right) \frac{1 - i\gamma L(s+2i\omega)g_s}{1 + i\gamma L(s+2i\omega)g_s}, \tag{A44}
\]

where \( \gamma = (w(r_+ - r_-))^{2l+1} \) \( g_s = \prod_{k=1}^l ((k^2 + 4\omega^2 - s^2 - 4is\omega)) \) and \( L_s \) is defined in Eq. (4.5) for \( l \neq \pm s \). Equation (A44) reduces to Eq. (A17) for \( s = 0 \).

If the mirror is close to the outer horizon, the field must vanish at the mirror surface. Using the small-\( r \) behavior of the near-region solution and setting the radial field to zero at \( z = z_0 \), it follows \( A = -B \frac{z_0^{s+2i\omega}}{2} \). The relation between the position of the mirror and the frequency of the wave is then given by Eq. (4.43).

**APPENDIX B: STAR BOMB FOR FAST ROTATING OBJECTS**

This appendix describes near-extremal star bombs by employing the analytic approximations of Refs. [52, 60]. The resonant frequencies for rapidly spinning BHs can be found by rewriting the Teukolsky equation in the Kerr incoming coordinate system \((v, r, \theta, \phi')\). This is obtained from the Beyer-Lindquist coordinates with the coordinate transformation \( dv = dt + (r^2 + a^2)dr/\Delta \). \( \phi' = \phi + adr/\Delta \). The radial equation for a field of spin \( s \) is [60]

\[
\Delta R''(r) + 2(s+1)(r-M) - iK \left[ R'(r) + \left[ 2(2s+1)i\omega + \lambda \right] R(r) = 0. \tag{B1}
\]

Defining \( x = (r - r_+)/r_+ \), \( \sigma = (r_+ - r_-)/r_+ \), \( \tau = M(\omega - m\Omega) \) and \( \omega' = \omega + \omega_0 \), Eq. (B1) reads

\[
x(x+\sigma)K''(x) - [2(2s+1)i\omega(x+1) + \lambda] K(x) - [2i\omega' (x+1) + 4i\omega' - 2(s+1)]K(x) + 4i\tau K(x) = 0. \tag{B2}
\]

In the far-region, \( x \gg \max(\sigma, \tau) \), Eq. (B2) reduces to

\[
x^2 K''(r) - (2i\omega' x^2 + 4i\omega' - 2(s+1)) K' + (2i\omega + i\lambda) K = 0. \tag{B3}
\]

Defining \( \delta^2 = 4\omega'^2 - (s+1)^2 - \lambda \), its most general solution is written in terms of confluent hypergeometric functions

\[
K(x) = \alpha x^{-s-\frac{1}{2}+i\phi} \times M \left( \frac{1}{2} + s + i\phi' + i\delta, 1 + 2i\delta, 2i\omega x \right) \tag{B4}
\]

\[
+ \beta(\delta \rightarrow -\delta),
\]

\[
\Delta R''(r) + 2(s+1)(r-M) - iK \left[ R'(r) + \left[ 2(2s+1)i\omega + \lambda \right] R(r) = 0. \tag{B1}
\]

\[
x(x+\sigma)K''(x) - [2(2s+1)i\omega(x+1) + \lambda] K(x) - [2i\omega' (x+1) + 4i\omega' - 2(s+1)]K(x) + 4i\tau K(x) = 0. \tag{B2}
\]

In the far-region, \( x \gg \max(\sigma, \tau) \), Eq. (B2) reduces to

\[
x^2 K''(r) - (2i\omega' x^2 + 4i\omega' - 2(s+1)) K' + (2i\omega + i\lambda) K = 0. \tag{B3}
\]

Defining \( \delta^2 = 4\omega'^2 - (s+1)^2 - \lambda \), its most general solution is written in terms of confluent hypergeometric functions

\[
K(x) = \alpha x^{-s-\frac{1}{2}+i\phi} \times M \left( \frac{1}{2} + s + i\phi' + i\delta, 1 + 2i\delta, 2i\omega x \right) \tag{B4}
\]

\[
+ \beta(\delta \rightarrow -\delta),
\]
where \((\delta \to -\delta)\) means “replace \(\delta\) by \(-\delta\) in the preceding term”. The large-\(x\) behavior of Eq. (B4) is

\[
R \sim x^{-1-2s} \left( \frac{\alpha \Gamma(1 + 2i\delta)(e^{i\pi/2}\lambda + 2i\omega + i\delta)}{\Gamma(1/2 - s - 2i\omega + i\delta)} \right) + \\
+ \frac{e^{2i\omega x}}{x^{1-4i\omega'}} \left( \frac{\alpha \Gamma(1 + 2i\delta)(2i\omega')^{-1/2 + 2i\omega' - i\delta}}{\Gamma(1/2 + s + 2i\omega' + i\delta)} \right) + \\
+ \beta(\delta \to -\delta) . \tag{B5}
\]

For near-extremal BHs, the term representing an incoming (outgoing) wave at infinity behaves as \(r^{-1-2s}\) \((e^{2i\omega x}/r)\). (See Table 1 of Ref. [60]). Thus the first and second terms in Eq. (B5) represent an incoming and an outgoing wave, respectively. Demanding no incoming waves at infinity implies

\[
\beta = -\frac{\Gamma(1 + 2i\delta)(1/2 - s - 2i\omega' - i\delta)}{\Gamma(1 - 2i\delta)(1/2 - s - 2i\omega' + i\delta)} (-2i\omega')^{-2i\delta} . \tag{B6}
\]

For small \(r\) Eq. (B5) reduces to

\[
R \sim \alpha x^{-s-1/2+2i\omega'+i\delta} + \beta x^{-s-1/2+2i\omega'-i\delta} . \tag{B7}
\]

In the near-region, Eq. (B2) reads

\[
x(x + \sigma)R''(x) - [x(4i\omega' - 2(s + 1)) - (s + 1)\sigma + 4i\tau]R'(x) + [2(2s + 1)i\omega' + \lambda]R(x) = 0 .
\]

Defining \(z = -x/\sigma\) the near-region radial wave equation becomes a standard hypergeometric equation. Its most general solution is

\[
R = A z^{1-c} F(a - c + 1, b - c + 1, 2 - c, z) + B F(a, b, c, z) , \tag{B8}
\]

where \(a = 1/2 + s - 2i\omega' + i\delta\), \(b = 1/2 + s - 2i\omega' - i\delta\), \(c = 1 + s + i\kappa\) and \(\kappa = -4\tau/\sigma\). At \(r \sim r_+\), where the mirror is located, the solution behaves as \(R \sim A z^{1-c} + B\). The large-\(r\) behavior of the near-region solution is

\[
R \sim \left( \frac{x}{\sigma} \right)^{-a} \left( A (-1)^{1-c} \frac{\Gamma(2 - c)\Gamma(b - a)}{\Gamma(b - c + 1)\Gamma(1 - a)} + B \frac{\Gamma(c)\Gamma(1 - a)}{\Gamma(b)\Gamma(1 - a)} \right) + \\
+ \left( \frac{x}{\sigma} \right)^{-b} \left( A (-1)^{1-c} \frac{\Gamma(2 - c)\Gamma(a - b)}{\Gamma(a - c + 1)\Gamma(1 - b)} + B \frac{\Gamma(c)\Gamma(a - b)}{\Gamma(a)\Gamma(c - b)} \right) , \tag{B9}
\]

or, in terms of the physical variables,

\[
R \sim \left( \frac{x}{\sigma} \right)^{-1/2 - s + 2i\omega' + i\delta} \times \\
\times \left[ \Gamma(2i\delta)(A (-1)^{s+i\kappa} R_1 + B R_2) \right] + (\delta \to -\delta) , \tag{B10}
\]

where

\[
R_1(\delta) = \frac{\Gamma(1 - s - i\kappa)}{\Gamma(1/2 - s + 2i\omega' + i\delta)\Gamma(1/2 - 2i\omega' + i\delta + i\kappa)} , \\
R_2(\delta) = \frac{\Gamma(1 + s + i\kappa)}{\Gamma(1/2 + s - 2i\omega' + i\delta)\Gamma(1/2 + 2i\omega' + i\delta + i\kappa)} , \\
R_3 = R_1(\delta) , \quad R_4 = R_2(-\delta) . \tag{B11}
\]

The matching in the overlapping region yields

\[
\alpha = \sigma^b \Gamma a - b(A (-1)^{1-c} R_1 + B R_2) , \\
\beta = \sigma^a \Gamma b - a(A (-1)^{1-c} R_3 + B R_4) . \tag{B12}
\]

Using Eq. (B6), it follows

\[
-\rho(-2i\omega')^{2i\delta} = A(-1)^{s+i\kappa} R_1 + B R_2 \overline{A(-1)^{s+i\kappa} R_3 + B R_4} . \tag{B13}
\]

For Dirichlet boundary conditions at the wall require \(z_0^{-s-i\kappa} = -B/A\). Finally, the relation between the position of the mirror and the frequency of the wave, Eq. (4.8), is obtained from Eq. (B14).


