

HYPERBOLICITY OF HIGH ORDER SYSTEMS OF EVOLUTION EQUATIONS

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Abstract. We study properties of evolution equations which are first order in time and arbitrary order in space (FTNS). Following Gundlach and Martín-García (2006) we define strong and symmetric hyperbolicity for FTNS systems and examine the relationship between these definitions, and the analogous concepts for first order systems. We demonstrate equivalence of the FTNS definition of strong hyperbolicity with the existence of a strongly hyperbolic first order reduction. We also demonstrate equivalence of the FTNS definition, up to $N = 4$, of symmetric hyperbolicity with the existence of a symmetric hyperbolic first order reduction.

Keywords: High-order PDEs; Strong hyperbolicity; Symmetric hyperbolicity

1. Introduction

Systems of partial differential equations admitting wave-like solutions are ubiquitous in both physics and applied mathematics. It is known that by restricting to the special case with at most first order derivatives the initial value problem of such systems can be classified algebraically with respect to its well-posedness. The crucial step in this classification is to check for strong hyperbolicity by analyzing the principal part, i.e. the derivative terms, of the evolution system [14,18].

The theory used to demonstrate this relies on pseudo-differential calculus [26]. By performing a pseudo-differential reduction to first order the basic method can also be applied to evolution systems with higher order derivatives, see for example [23,11].

For the initial boundary value problem the theory is not so complete. The simplest approach for first order systems is to check for a stronger condition, called

symmetric hyperbolicity. With carefully chosen boundary conditions it can be used to identify a well-posed initial boundary value problem [14,18]. If the evolution system is not symmetric hyperbolic there is still hope to demonstrate well-posedness, e.g. by employing the Laplace-Fourier method [17,2,22,25], which unfortunately does not apply to arbitrary strongly hyperbolic evolution systems.

We study strong and symmetric hyperbolicity for a special class of higher order evolution equations. Hyperbolicity of higher order systems was studied before in a different context, see e.g. [3,27,7]. The equations of interest here are linear constant coefficient, first order in time and arbitrary order in space (FTNS). They admit a reduction to first order for which standard definitions of hyperbolicity are applicable.

Reductions to first order are obtained by introducing new variables for all but the highest order derivatives [8], which is a common approach in numerical relativity [15,24,4,20]. In this way the known, first order definitions of hyperbolicity can be applied, and powerful numerical methods are available in the construction of approximate solutions [14,18,13].

However, making the first order reduction raises questions, e.g. about the number of constraints to impose and the size of the approximation error [19,6]. For practical applications it also incurs a cost. The memory footprint of any numerical approximation method increases hugely due to the auxiliary variables.

The question we address here is whether or not we can characterize hyperbolicity of FTNS systems without making a differential or pseudo-differential reduction to first order. The idea is to establish when “good” reductions of either type can be made. For the important case of second order in space systems this question was already answered satisfactorily in the affirmative by Gundlach and Martín-García [11], see also [10,9,12] for applications of these ideas. The present work is the extension of those calculations to first order in time, higher order in space systems. The generalization here will be useful in analyzing higher derivative systems. A more abstract treatment of evolution systems can be found in [5].

We propose definitions of strong and symmetric hyperbolicity for FTNS systems without reference to any first order system. This enables us to demonstrate equivalence of FTNS strong hyperbolicity with the existence of an iterative first order reduction, either differential or pseudo-differential, that is strongly hyperbolic in the sense of first order systems.

We also find that if a higher order system has a symmetric hyperbolic first order reduction then the equations must satisfy the FTNS definition of symmetric hyperbolicity. Conversely, for systems containing up to fourth order spatial derivatives, we show that the new definition of symmetric hyperbolicity is also sufficient for the existence of a symmetric hyperbolic first order reduction.

The first order reduction used in this case is a direct, not iterative method, i.e. it differs from the one applied in the proofs concerning strong hyperbolicity. The iterative, order-by-order reduction is not appropriate for symmetric hyperbolicity. We give an explicit example of a third order symmetric hyperbolic evolution problem without an appropriate second order symmetric hyperbolic reduction, but which

does have a direct reduction to first order that is symmetric hyperbolic.

The Laplace-Fourier method, which can be used to prove well-posedness of initial boundary value problems is not considered here. Higher order derivative evolution systems can be treated by this technique (see for example [16]), because it once again relies on pseudo-differential calculus.

The paper is structured as follows. In section 2 we review the definitions of strong and symmetric hyperbolicity for first order in time, second order in space systems. For pedagogical purposes, in section 3, we explicitly present the special case of the extension of the theory to first order in time, third order in space systems. Then we provide a general formulation of first order in time, N -th order in space systems in section 4. In section 5 we discuss strong hyperbolicity using an iterative reduction procedure. In section 6 definitions for symmetric hyperbolicity are given for the higher order system without reduction. The relationship between the definitions is then investigated using a direct reduction to first order. We conclude in section 7. Appendix A-Appendix D contain various details that are relegated from the main body of the text.

2. Basic notions of hyperbolicity

In this article we consider a special class of linear systems of partial differential equations with constant coefficients. We are mainly interested in questions about the well-posedness of initial (boundary) value problems.

Well-posedness: An initial (boundary) value problem is called well-posed if there is a unique solution that depends continuously, in some appropriate norm, on the choice of initial data.

Second order systems: The class of partial differential equations under consideration is a generalization of the first order in time, second order in space systems analyzed in [11,10,9]. We start with a short summary of that work. Consider first order in time, second order in space systems of the form

$$\partial_t \tilde{u} = (A^u_u)^i \partial_i \tilde{u} + A^u_v \tilde{v} + S_u, \quad (2.1a)$$

$$\partial_t \tilde{v} = (A^v_u)^{ij} \partial_i \partial_j \tilde{u} + (A^v_v)^i \partial_i \tilde{v} + S_v. \quad (2.1b)$$

where we have absorbed all non-principal terms into the source functions S . They have the form

$$S_u = \alpha_1 \tilde{u} + f_u, \quad (2.2a)$$

$$S_v = \alpha_2^i \partial_i \tilde{u} + \alpha_3 \tilde{u} + \alpha_4 \tilde{v} + f_v, \quad (2.2b)$$

where f_u and f_v do not depend on \tilde{u} or \tilde{v} and the α_i are constant coefficient matrices.

Principal part: The *principal part* of the system (2.1) is

$$\partial_t \tilde{u} \simeq (A^u_u)^i \partial_i \tilde{u} + A^u_v \tilde{v}, \quad (2.3a)$$

$$\partial_t \tilde{v} \simeq (A^v_u)^{ij} \partial_i \partial_j \tilde{u} + (A^v_v)^i \partial_i \tilde{v} \quad (2.3b)$$

where \simeq denotes equality up to non-principal terms. We denote the matrix

$$\mathcal{A}_{2i}^{p,j} = \begin{pmatrix} (A^u_u)^j \delta^p_i & A^u_v \delta^p_i \\ (A^v_u)^{pj} & (A^v_v)^p \end{pmatrix} \quad (2.4)$$

the *principal matrix* of the system (2.1). For a fixed spatial vector s^i the *principal symbol* of the system (2.1) is

$$P_2^s = \begin{pmatrix} (A^u_u)^i s_i & A^u_v \\ (A^v_u)^{ij} s_i s_j & (A^v_v)^i s_i \end{pmatrix}. \quad (2.5)$$

Note that with

$$S_i := \begin{pmatrix} s_i & 0 \\ 0 & \mathbf{1} \end{pmatrix} \quad (2.6)$$

one obtains the principal symbol from the principal matrix by the contraction

$$P_2^s = S^i \mathcal{A}_{2i}^{p,j} S_j s_p. \quad (2.7)$$

Furthermore the equations of motion for the variables

$$\tilde{u}_i := \begin{pmatrix} \partial_i \tilde{u} \\ \tilde{v} \end{pmatrix} \quad (2.8)$$

are up to non principal terms

$$\partial_t \tilde{u}_i \simeq \mathcal{A}_{2i}^{p,j} \partial_p \tilde{u}_j. \quad (2.9)$$

Strong hyperbolicity: Following [11,25,10,9] the system (2.1) is called strongly hyperbolic if there exist a constant $M_2 > 0$ and a family of hermitian matrices $H_2(s)$ such that

$$H_2(s) P_2^s = (P_2^s)^\dagger H_2(s), \quad (2.10a)$$

$$M_2^{-1} I \leq H_2(s) \leq M_2 I, \quad (2.10b)$$

where we used the standard inequality for hermitian matrices

$$A \leq B \iff v^\dagger A v \leq v^\dagger B v \quad \forall v. \quad (2.11)$$

It is a necessary and sufficient condition for well-posedness of the initial value problem. This definition is furthermore equivalent to the existence of a fully first order reduction of (2.1) which satisfies the standard definition of strong hyperbolicity for first order systems.

Note that this is not quite equivalent to the definition given in [11,10,9], where it is required that the principal symbol has real eigenvalues and a complete set of eigenvectors that depend continuously on s .

What can be shown [25,18] is that (2.10) is equivalent to the existence of a constant $K_2 > 0$ and a family of matrices $T_2(s)$ such that

$$T_2(s)^{-1} P_2^s T_2(s) = \Lambda(s), \quad K_2^{-1} \leq \|T_2(s)\| \leq K_2, \quad (2.12)$$

with a real, diagonal matrix $\Lambda(s)$ and the standard spectral norm $\|\cdot\|$.

In view of example 12 in [25], the continuity of $T_2(s)$ required in [11,10,9] is sufficient to guarantee the existence of K_2 , but not necessary. Fortunately despite the continuity condition being slightly too restrictive, the construction of first order reductions with the approach of [11] is unaltered if we instead require (2.12). Our treatment of strong hyperbolicity for FTNS systems is therefore the natural generalization of [11].

Symmetric hyperbolicity: For the analysis of the initial *boundary* value problem the stronger notion of symmetric hyperbolicity is desirable. It guarantees the existence of a conserved energy in the principal part and allows the construction of boundary conditions such that the initial boundary value problem is well posed. A Hermitian matrix

$$H_2^{ij} = \begin{pmatrix} H_{11}^{ij} & H_{12}^i \\ H_{12}^{\dagger j} & H_{22} \end{pmatrix}, \quad (2.13)$$

independent of s^i , such that the matrix

$$S_i H_2^{ij} A_{2j}^p s_p S_k, \quad (2.14)$$

is Hermitian for every spatial vector s^i is called a candidate symmetrizer. The system (2.1) is called symmetric hyperbolic if there exists a positive definite candidate symmetrizer. The aforementioned conserved energy is

$$E = \int d^3x \epsilon = \int d^3x \tilde{u}_i^\dagger H_2^{ij} \tilde{u}_j. \quad (2.15)$$

It can be shown that $\partial_t E \simeq 0$ if (2.14) is Hermitian [11].

3. Third order systems

Before starting with the generalization to arbitrary order we discuss third order systems here. In [11] Gundlach and Martín-García give different possible definitions of hyperbolicity of second order systems. They showed that these definitions are equivalent to the existence of a first order reduction with the same level of hyperbolicity. We follow a similar approach here.

3.1. Definition of third order systems

FT3S systems: We consider first order in time, third order in space (FT3S) systems of the form

$$\begin{aligned}\partial_t u &= (A^u_u)^i \partial_i u + (A^u_v)v + (B^u_{1u})u + s^u, \\ \partial_t v &= (A^v_u)^{ij} \partial_i \partial_j u + (A^v_v)^i \partial_i v + (A^v_w)w + (B^v_{1u})^i \partial_i u + (B^v_{2u})u + (B^v_{1v})v + s^v, \\ \partial_t w &= (A^w_u)^{ijk} \partial_i \partial_j \partial_k u + (A^w_v)^{ij} \partial_i \partial_j v + (A^w_w)^i \partial_i w + (B^w_{1u})^{ij} \partial_i \partial_j u + (B^w_{2u})^i \partial_i u \\ &\quad + (B^w_{3u})u + (B^w_{1v})^i \partial_i v + (B^w_{2v})v + (B^w_{1w})w + s^w,\end{aligned}\tag{3.1}$$

where s^u , s^v and s^w are arbitrary source terms that do not depend on u , v or w . In analogy to the second order case we define the principal part of that system as

$$\begin{aligned}\partial_t u &\simeq (A^u_u)^i \partial_i u + (A^u_v)v, \\ \partial_t v &\simeq (A^v_u)^{ij} \partial_i \partial_j u + (A^v_v)^i \partial_i v + (A^v_w)w, \\ \partial_t w &\simeq (A^w_u)^{ijk} \partial_i \partial_j \partial_k u + (A^w_v)^{ij} \partial_i \partial_j v + (A^w_w)^i \partial_i w,\end{aligned}\tag{3.2}$$

where as before \simeq denotes equality up to non principal terms. As the principal matrix of the system (3.1) we define

$$A_{3kl}^{p\ mn} = \begin{pmatrix} \delta_{(k}^p \delta_{l)}^m (A^u_u)^n & \delta_{(k}^p \delta_{l)}^m (A^u_v) & 0 \\ \delta_k^p (A^v_u)^{mn} & \delta_k^p (A^v_v)^m & \delta_k^p (A^v_w) \\ (A^w_u)^{pmn} & (A^w_v)^{pm} & (A^w_w)^p \end{pmatrix}\tag{3.3}$$

and the principal symbol is

$$P_3^s = S^{ij} A_{3ij}^{p\ kl} S_{kl} s_p\tag{3.4}$$

where

$$S_{ij} = \begin{pmatrix} s_i s_j & 0 & 0 \\ 0 & s_i & 0 \\ 0 & 0 & 1 \end{pmatrix}.\tag{3.5}$$

3.2. Reduction to second order

Reduction variables: We are going to define strong hyperbolicity of FT3S systems by referring to strong hyperbolicity of second order systems. Here we define what we mean by a reduction of the FT3S system (3.1) to second order. We introduce a vector of reduction variables d_a . The reduction variables eventually replace the spatial derivatives of the fields u in the reduced system:

$$d_a = \partial_a u.\tag{3.6}$$

We use lower case letters from the beginning of the Latin alphabet as derivative indices without further meaning. In what follows their use simply helps to identify indices which belong to d which makes it simpler to work with the principal matrix of the second order reduction.

Unmodified evolution equations: The aim is now to include the d_a as independent variables in a first order in time second order in space (FT2S) system. Therefore an evolution equation for these variables is needed which must be consistent with (3.6). One gets this equation e.g. by taking the spatial derivative of the evolution equation for u :

$$\partial_t d_a = (A^u_u)^j \partial_a d_j + (A^u_v) \partial_a v + (B^u_{1u}) \partial_a u + \partial_a s^u. \quad (3.7)$$

In this equation $\partial_a s^u$ does not depend on the variables u, v, w or d_a and hence can be considered as a given source function.

Auxiliary constraints: Obviously the system composed of (3.1) and (3.7) is not second order. However, one can get rid of the higher order terms by adding linear combinations of the following functions and their derivatives to the right hand sides

$$\begin{aligned} c_a &:= \partial_a u - d_a, & c_{ia} &:= \frac{1}{2} (\partial_i d_a - \partial_a d_i), \\ c_{ija} &:= \partial_i \partial_j d_a - \partial_{(i} \partial_{j} d_{a)}. \end{aligned} \quad (3.8)$$

These functions vanish when equation (3.6) is satisfied. We will show that their evolution system is closed for the FT2S systems that we consider here. The functions c are denoted *auxiliary constraints*. Furthermore the c_{ija} can be written as a linear combination of derivatives of the c_{ia} :

$$c_{ija} = 2/3 \partial_i c_{ja} + 2/3 \partial_j c_{ia}. \quad (3.9)$$

Therefore their addition to the right hand sides is already covered by the addition of derivatives of the c_{ia} . We do not consider the c_{ija} separately.

Reduced system: FT2S systems which are obtained in that way have the form

$$\begin{aligned} \partial_t u &= (A^u_u)^i \partial_i u + (A^u_v) v + (B^u_{1u}) u + s^u + (D^u)^a c_a + (\bar{D}^u)^{ia} c_{ia} \\ \partial_t d_a &= (B^u_{1u}) \partial_a u + (A^u_u)^b \partial_a d_b + (A^u_v) \partial_a v + \partial_a s + (D^u)_a{}^b c_b + (\bar{D}^u)_a{}^{kb} c_{kb}, \\ \partial_t v &= (B^v_{1u})^i \partial_i u + (A^v_u)^{ia} \partial_i d_a + (A^v_v)^i \partial_i v + (A^v_w) w + (B^v_{2u}) u + (B^v_{1v}) v + s^v \\ &\quad + (D^v)^a c_a + (\bar{D}^v)^{ia} c_{ia}, \\ \partial_t w &= (B^w_{1u})^{ij} \partial_i \partial_j u + (A^w_u)^{ija} \partial_i \partial_j d_a + (A^w_v)^{ij} \partial_i \partial_j v + (A^w_w)^i \partial_i w + (B^w_{2u})^i \partial_i u \\ &\quad + (B^w_{3u}) u + (B^w_{1v})^i \partial_i v + (B^w_{2v}) v + (B^w_{1w}) w + (D^w)^{ka} \partial_k c_a \\ &\quad + (\bar{D}^w)^{kja} \partial_k c_{ja}. \end{aligned} \quad (3.10)$$

We denote the constant matrices D and \bar{D} the *reduction parameters*. Since c_{jb} is antisymmetric we can assume without loss of generality

$$(\bar{D}^u)^{ia} = -(\bar{D}^u)^{ai}, \quad \bar{D}_a{}^{kb} = -\bar{D}_a{}^{bk}, \quad (3.11a)$$

$$(\bar{D}^v)^{ia} = -(\bar{D}^v)^{ai}, \quad (\bar{D}^w)^{kja} = -(\bar{D}^w)^{kaj}. \quad (3.11b)$$

Definition 1. We call a first order in time, second order in space system of the form (3.10) an FT2S reduction of the first order in time, third order in space system (3.1).

This definition of a reduction to second order is quite restrictive, one may think of other definitions that are satisfied by more second order systems. Indeed one finds that it is too restrictive to be used in a definition of symmetric hyperbolicity for FT3S systems. This topic is discussed in Appendix C.

Auxiliary constraint evolution: For every FT2S reduction of (3.1), provided that the reduction constraints are satisfied, one can show that there is a relationship between solutions of the two systems.

Lemma 1. *If the system (3.10) is an FT2S reduction of (3.1) and (u, d_a, v, w) is a solution of (3.10) with vanishing auxiliary constraints (3.8) then (u, v, w) is a solution of the FT3S system (3.1). Moreover, if (u, v, w) is a solution of the FT3S system (3.1) and the system (3.10) is an FT2S reduction of (3.1) then $(u, \partial_a u, v, w)$ is a solution of the FT2S system (3.10) with vanishing auxiliary constraints (3.8).*

Proof. By inserting the subset (u, v, w) of the FT2S solution into the FT3S system one can easily check that these functions satisfy (3.1), because the auxiliary constraints (3.8) vanish by assumption. Moreover, if (u, v, w) is a solution of (3.1) then one can insert $(u, \partial_a u, v, w)$ into the system (3.10) to see that it is a solution.

The reason for this being that the auxiliary constraint evolution system is closed:

$$\begin{aligned} \partial_t c_a &= \partial_t d_a - \partial_a \partial_t u \\ &= (A^u{}_u)^b \partial_a c_b + ((D)_a{}^b - (D^u)^b \partial_a) c_b + ((\bar{D})_a{}^{kb} - (\bar{D}^u)^{kb} \partial_a) c_{kb}, \end{aligned} \quad (3.12a)$$

$$\begin{aligned} \partial_t c_{ia} &= \partial_i \partial_t d_a - \partial_a \partial_t d_i \\ &= (D)_a{}^b \partial_i c_b + (\bar{D})_a{}^{kb} \partial_i c_{kb} - (D)_i{}^b \partial_a c_b - (\bar{D})_i{}^{kb} \partial_a c_{kb}, \\ \partial_t c_{ija} &= 2/3 \partial_j \partial_t c_{ia} + 2/3 \partial_a \partial_t c_{ij}. \end{aligned} \quad (3.12b)$$

It is straightforward to check that $(u, \partial_a u, v, w)$ solves (3.8). \square

Principal part of the FT2S reduction: According to the definitions given in section 2 the principal part of the FT2S reduction (3.10) is

$$\begin{aligned} \partial_t u &\simeq (A^u{}_u)^i \partial_i u + (D^u)^a \partial_a u + (\bar{D}^u)^{ia} \partial_i d_a, \\ \partial_t d_a &\simeq (B^u{}_{1u}) \partial_a u + (D)_a{}^b \partial_b u + (A^u{}_u)^b \partial_a d_b + (\bar{D})_a{}^{kb} \partial_k d_b + (A^u{}_v) \partial_a v, \\ \partial_t v &\simeq (B^v{}_{1u})^i \partial_i u + (D^v)^a \partial_a u + (A^v{}_u)^{ia} \partial_i d_a + (\bar{D}^v)^{ia} \partial_i d_a + (A^v{}_v)^i \partial_i v + (A^v{}_w) w, \\ \partial_t w &\simeq (B^w{}_{1u})^{ij} \partial_i \partial_j u + (D^w)^{ka} \partial_k \partial_a u + (A^w{}_u)^{ija} \partial_i \partial_j d_a + (\bar{D}^w)^{kja} \partial_k \partial_j d_a \\ &\quad + (A^w{}_v)^{ij} \partial_i \partial_j v + (A^w{}_w)^i \partial_i w \end{aligned} \quad (3.13)$$

and the principal matrix is

$$\mathcal{A}_{2^i a}^p{}^j{}^b = \begin{pmatrix} \delta_i^p ((A^u{}_u)^j + (D^u)^j) & \delta_i^p (\bar{D}^u)^{jb} & 0 & 0 \\ \delta_i^p ((B^u{}_{1u}) \delta_a^j + (D)_a{}^j) & \delta_i^p ((A^u{}_u)^b \delta_a^j + (\bar{D})_a{}^{jb}) & (A^u{}_v) \delta_i^p \delta_a^j & 0 \\ \delta_i^p ((B^v{}_{1u})^j + (D^v)^j) & \delta_i^p ((A^v{}_u)^{jb} + (\bar{D}^v)^{jb}) & \delta_i^p (A^v{}_v)^j & \delta_i^p (A^v{}_w) \\ (B^w{}_{1u})^{pj} + (D^w)^{pj} & (A^w{}_u)^{pj} + (\bar{D}^w)^{pj} & (A^w{}_v)^{pj} & (A^w{}_w)^p \end{pmatrix}. \quad (3.14)$$

3.3. Strong hyperbolicity

Definitions of strong hyperbolicity: We show that the following definitions of third order strong hyperbolicity are equivalent

Definition 2a. *The FT3S system (3.1) is called FT2S strongly hyperbolic if there exists an FT2S reduction (3.10) which is strongly hyperbolic in the sense described in section 2.*

Definition 2b. *The FT3S system (3.1) is called FT3S strongly hyperbolic if there exist a constant $M_3 > 0$ and a family of hermitian matrices $H_3(s)$ such that*

$$H_3(s)P_3^s = (P_3^s)^\dagger H_3(s), \quad M_3^{-1} I \leq H_3(s) \leq M_3 I, \quad (3.15)$$

where the matrix inequality is understood in the standard sense (2.11).

With this one can apply an iterative procedure which reduces strong hyperbolicity of FT3S systems to strong hyperbolicity of fully first order systems. First one reduces the FT3S system to second order and after that the resulting FT2S system to a fully first order system by applying the work of Gundlach and Martín-García [11], $\text{FT3S} \rightarrow \text{FT2S} \rightarrow \text{FT1S}$.

A third possible definition of strong hyperbolicity employs a pseudo-differential reduction. One finds that this definition is very similar to our definition 2b. We discuss the topic in Appendix A for systems of arbitrary order.

FT2S strong hyperbolicity \Rightarrow FT3S strong hyperbolicity: In the proof that definition 2a implies 2b we start with a 2+1 decomposition of the reduction variable d_a . With the orthogonal projector, q_a^A , of the given vector s we decompose $d_a = q_a^B d_B + s_a d_s$, where $d_B s^B = 0$. Partitioning the state vector as (u, d_A, d_s, v, w) the FT2S principal symbol $P_{2A}^s{}^B$ is

$$P_{2A}^s{}^B = \begin{pmatrix} \tilde{X}_A^B & 0 \\ \tilde{Y}^B & P_3^s \end{pmatrix}, \quad (3.16)$$

where

$$\tilde{X}_A^B = \begin{pmatrix} ((A^u_u)^j + (D^u)^j) s_j & (\bar{D}^u)^{jb} s_j q_b^B \\ (D)_a^j q_A^a s_j & (\bar{D})_a^{jb} q_A^a s_j q_b^B \end{pmatrix} \quad (3.17)$$

and

$$\tilde{Y}^B = \begin{pmatrix} (B^u_{1u}) + (D)_a^j s^a s_j & ((A^u_u)^b + (\bar{D})_a^{jb} s^a s_j) q_b^B \\ ((B^v_{1u})^j + (D^v)^j) s_j & ((A^v_u)^{jb} + (\bar{D}^v)^{jb}) s_j q_b^B \\ ((B^w_{1u})^{pj} + (D^w)^{pj}) s_p s_j & ((A^w_u)^{pj b} + (\bar{D}^w)^{pj b}) s_p s_j q_b^B \end{pmatrix}. \quad (3.18)$$

There we used that the \bar{D} are antisymmetric in the last two indices. That is, if one contracts both indices with s then the result vanishes.

The assumption that (3.1) is FT2S strongly hyperbolic means that there exist a constant M_2 and a family of matrices $H_2(s)^{AB}$ such that

$$H_2(s)^{AB} P_{2B}^s{}^C = (P_{2B}^s{}^A)^\dagger H_2(s)^{BC}, \quad (3.19a)$$

$$M_2^{-1} I^{AB} \leq H_2(s)^{AB} \leq M_2 I^{AB}, \quad (3.19b)$$

where I^{AB} is the appropriate identity matrix.

We decompose $H_2(s)^{AB}$ in a way compatible to the decomposition in (3.16):

$$H_2(s)^{AB} = \begin{pmatrix} H_{11}(s)^{AB} & H_{12}(s)^A \\ H_{12}(s)^\dagger{}^B & H_{22}(s) \end{pmatrix}, \quad (3.20)$$

and find

$$H_2(s)^{AB} P_{2B}^s{}^C = \begin{pmatrix} H_{11}(s)^{AB} \tilde{X}_B^C + H_{12}(s)^A \tilde{Y}^C & H_{12}(s)^A P_3^s \\ H_{12}(s)^\dagger{}^B \tilde{X}_B^C + H_{22}(s) \tilde{Y}^C & H_{22}(s) P_3^s \end{pmatrix}. \quad (3.21)$$

Looking at the lower right block of this expression equation (3.19a) implies

$$H_{22}(s) P_3^s = (P_3^s)^\dagger H_{22}(s). \quad (3.22)$$

Furthermore we have obviously $H_{22}(s) = H_{22}(s)^\dagger$ and

$$M_2^{-1} v^\dagger v \leq v^\dagger H_{22}(s) v = (0, v^\dagger) H_2(s)^{AB} (0, v^\dagger)^\dagger \leq M_2 v^\dagger v \quad \forall v, \quad (3.23)$$

because (3.19b) is satisfied by assumption.

Hence, the matrix $H_3(s) := H_{22}(s)$ satisfies (3.15) and FT3S strong hyperbolicity of (3.1) is shown.

FT3S strong hyperbolicity \Rightarrow FT2S strong hyperbolicity: For the reverse direction we need to choose the reduction parameters appropriately. One can check easily that the first row and column of (3.16) vanish with the choice

$$(D^u)^j = -(A^u{}_u)^j, \quad (D)_a{}^j = -(B^u{}_{1u}) \delta_a^j, \quad (D^v)^j = -(B^v{}_{1u})^j, \quad (3.24a)$$

$$(D^w)^{pj} = -(B^w{}_{1u})^{pj}, \quad (\bar{D}^u)^{jb} = 0. \quad (3.24b)$$

We call (3.24) the *partial choice* of reduction parameters. Under the partial choice $P_{2A}^s{}^B$ has the following lower block triangular form,

$$P_{2A}^s{}^B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & X_A^B & 0 \\ 0 & Y^B & P_3^s \end{pmatrix}, \quad (3.25)$$

where

$$X_A^B = (\bar{D})_a{}^{jb} q_A^a s_j q_b^B, \quad (3.26a)$$

$$Y^B = \begin{pmatrix} ((A^u{}_u)^b + (\bar{D})_a{}^{jb} s^a s_j) q_b^B \\ ((A^v{}_u)^{jb} + (\bar{D}^v)^{jb}) s_j q_b^B \\ ((A^w{}_u)^{pj} + (\bar{D}^w)^{pj}) s_p s_j q_b^B \end{pmatrix}. \quad (3.26b)$$

As mentioned in section 2, definition 2a is equivalent to the existence of a constant K_2 and a family of matrices $T_2(s)_A^B$ with

$$K_2^{-1} \leq \|T_2(s)_A^B\| \leq K_2 \quad (3.27)$$

such that $T_2(s)^{-1}_A^B P_{2B}^s T_2(s)_C^D$ is real and diagonal. Here we show this property instead of the original definition.

Following [11] we choose the reduction parameters such that X_A^B is diagonalizable:

$$(\bar{D})_a^{jb} = i\lambda \varepsilon_a^{jb} \quad (3.28)$$

with $\lambda \in \mathbb{R}$ and ε_a^{jb} the Levi-Civita symbol. The eigenvalues of X_A^B become $\pm\lambda$. They are independent of s and the eigenvalues of P_{2A}^s are the union of the eigenvalues of P_3^s and $\pm\lambda$.

Using that P_3^s is bounded, because it is a sum of products of bounded matrices:

$$\|P_3^s\| = \|S^{ij} \mathcal{A}^p_{ij}{}^{kl} S_{kl} s_p\| \leq \|S_{ij}\| \|\mathcal{A}^p_{ij}{}^{kl}\| \|S_{kl}\| \|s_p\|, \quad (3.29)$$

we choose λ larger than all eigenvalues of P_3^s . Together with the assumption that (3.1) is FT3S strongly hyperbolic, i.e. that P_3^s is diagonalizable, this choice of λ makes P_{2A}^s diagonalizable as well.

The corresponding similarity transformation can be constructed from the eigenvectors of P_{2A}^s . One finds that given an eigenvector, v , of P_3^s with eigenvalue α and an eigenvector, w_B , of X_A^B then

$$P_{2A}^s \begin{pmatrix} 0 \\ v \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad P_{2A}^s \begin{pmatrix} w_B \\ w \end{pmatrix} = \lambda \begin{pmatrix} w_B \\ w \end{pmatrix},$$

where we used $w := (\lambda - P_3^s)^{-1} Y^B w_B$, which exists, because λ does not coincide with an eigenvalue of P_3^s .

Now, a matrix which makes $T_2(s)^{-1}_A^B P_{2B}^s T_2(s)_C^D$ diagonal (and real) is

$$T_2(s)_A^B = \begin{pmatrix} W_A^B & 0 \\ (\lambda - P_3^s)^{-1} Y^A W_A^B & T_3(s) \end{pmatrix}, \quad (3.30)$$

where $T_3(s)$ and W_A^B diagonalize P_3^s and X_A^B respectively. The inverse of $T_2(s)_A^B$ is

$$T_2(s)^{-1}_A^B = \begin{pmatrix} W^{-1}_A{}^B & 0 \\ -T_3(s)^{-1} (\lambda - P_3^s)^{-1} Y^B & T_3(s)^{-1} \end{pmatrix}.$$

Both, $T_2(s)_A^B$ and its inverse are bounded, because on the one hand $T_3(s)$ and $T_3(s)^{-1}$ are bounded by the assumption (3.15) and we have chosen λ such that $(\lambda - P_3^s)^{-1}$ is bounded as well.

Hence, we get that there exists a constant $K_2 > 0$ such that

$$K_2^{-1} \leq \|T_2(s)_A^B\| \leq K_2, \quad (3.31)$$

which shows that FT3S strong hyperbolicity implies FT2S strong hyperbolicity. \square

3.4. Why two different reductions?

Failure of the iterative procedure for symmetric hyperbolicity: Symmetric hyperbolicity relies fundamentally on conserved quantities (we will discuss the details of FT3S conservation equations in section 3.6). Hence, in order to deal with symmetric hyperbolicity for the second order reductions, which were used to handle strong hyperbolicity, we need to construct a reduction with a conserved quantity that is associated to the given FT3S symmetrizer. However, one finds that there are FT3S systems with a conserved energy for which no FT2S reduction with the same conserved quantity exists. We give such a counterexample in C.2 and the reason why an FT3S conserved quantity does not allow the construction of a conserved quantity for an FT2S reduction is discussed in Appendix C.

Discussion: This situation differs from the case of reductions of FT2S systems to first order. There every FT2S symmetrizer implies an FT1S conserved energy. In [11] this was the basis of the proof that for every symmetric hyperbolic FT2S system there exists a symmetric hyperbolic first order reduction. Thus, we cannot use the iterative procedure to prove existence of symmetric hyperbolic lower order reductions. In order to avoid this problem we employ a direct reduction to first order (described in section 3.5) and construct a conserved quantity for the first order system.

Why not always use the direct reduction? Conversely, one may also think of using the direct first order reduction to show statements about strong hyperbolicity. There the problem is that the proofs rely on the choice of reduction parameters (3.28). For direct first order reductions the structure of reduction parameters changes completely, and we did not find a choice that shows existence of a strongly hyperbolic direct first order reduction. So we use one class of reductions for proofs about strong hyperbolicity, namely reductions from FT3S to FT2S, and another class for proofs on symmetric hyperbolicity, namely reductions from FT3S to FT1S.

3.5. Direct reduction to first order

Reduction variables: In analogy to the construction of FT2S reductions of the FT3S system (3.1) we now define *direct first order reductions* of (3.1). We also use the terminology *direct FT1S reduction*. We define reduction variables

$$d_i^u = \partial_i u, \quad d_{ij}^u = \partial_i d_j^u, \quad d_i^v = \partial_i v. \quad (3.32)$$

The equations of motion which one derives from these definitions are

$$\partial_t d_i^u = (A^u_u)^j \partial_i \partial_j u + (A^u_v) \partial_i v + (B^u_{1u}) \partial_i u + \partial_i s^u, \quad (3.33a)$$

$$\partial_t d_{ij}^u = (A^u_u)^k \partial_i \partial_j \partial_k u + (A^u_v) \partial_i \partial_j v + (B^u_{1u}) \partial_i \partial_j u + \partial_i \partial_j s^u, \quad (3.33b)$$

$$\begin{aligned} \partial_t d_i^v &= (A^v_u)^{jk} \partial_i \partial_j \partial_k u + (A^v_v)^j \partial_i \partial_j v + (A^v_w) \partial_i w + (B^v_{1u})^j \partial_i \partial_j u + (B^v_{2u}) \partial_i u \\ &\quad + (B^v_{1v}) \partial_i v + \partial_i s^v. \end{aligned} \quad (3.33c)$$

Auxiliary constraints: They are subject to the first order auxiliary constraints

$$\begin{aligned} c_i^u &= \partial_i u - d_i^u, & \bar{c}_{ij}^u &= \partial_i d_j^u - \partial_{(i} d_{j)}^u, & c_{ij}^u &= \partial_{(i} d_{j)}^u - d_{ij}^u, \\ \bar{c}_{ijk}^u &= \partial_i d_{jk}^u - \partial_{(i} d_{jk)}^u, & c_i^v &= \partial_i v - d_i^v, & \bar{c}_{ij}^v &= \partial_i d_j^v - \partial_{(i} d_{j)}^v. \end{aligned} \quad (3.34)$$

We call a first order system which is composed of equations (3.1) and (3.33) with additions of linear combinations of the auxiliary constraints (3.34) and their derivatives to the right hand sides a *direct first order reduction* of the FT3S system (3.1).

Reduction: Note that we allow additions of derivatives of the auxiliary constraints, but it is not possible to add arbitrary derivatives, because the final system must be first order. The constraint additions are used to cancel the higher order terms in (3.1) and (3.33).

As in section 3.2 one can show that there is a one-to-one relation between solutions of (3.1) and the solutions of first order reductions which satisfy the auxiliary constraints. The reason is again that the auxiliary constraint evolution system in the first order reduction is closed. We show this step for arbitrary spatial derivative order in section 6.1 and Appendix D.

The principal part of a first order reduction of (3.1) has the form

$$\begin{aligned} \partial_t u &\simeq (D^u_u)^k c_k^u + (D^u_u)^{kl} c_{kl}^u + (D^u_v)^k c_k^v + (\bar{D}^u_u)^{kl} \bar{c}_{kl}^u + (\bar{D}^u_u)^{klm} \bar{c}_{klm}^u \\ &\quad + (\bar{D}^u_v)^{kl} \bar{c}_{kl}^v, \end{aligned} \quad (3.35a)$$

$$\begin{aligned} \partial_t d_i^u &\simeq (D^u_u)_i^k c_k^u + (D^u_u)_i^{kl} c_{kl}^u + (D^u_v)_i^k c_k^v + (\bar{D}^u_u)_i^{kl} \bar{c}_{kl}^u + (\bar{D}^u_u)_i^{klm} \bar{c}_{klm}^u \\ &\quad + (\bar{D}^u_v)_i^{kl} \bar{c}_{kl}^v, \end{aligned} \quad (3.35b)$$

$$\begin{aligned} \partial_t v &\simeq (D^v_u)^k c_k^u + (D^v_u)^{kl} c_{kl}^u + (D^v_v)^k c_k^v + (\bar{D}^v_u)^{kl} \bar{c}_{kl}^u + (\bar{D}^v_u)^{klm} \bar{c}_{klm}^u \\ &\quad + (\bar{D}^v_v)^{kl} \bar{c}_{kl}^v, \end{aligned} \quad (3.35c)$$

$$\begin{aligned} \partial_t d_{ij}^u &\simeq (A^u_u)^k \partial_{(i} d_{j)k}^u + (A^u_v) \partial_{(i} d_{j)}^v + (D^u_u)_{ij}^k c_k^u + (D^u_u)_{ij}^{kl} c_{kl}^u + (D^u_v)_{ij}^k c_k^v \\ &\quad + (\bar{D}^u_u)_{ij}^{kl} \bar{c}_{kl}^u + (\bar{D}^u_u)_{ij}^{klm} \bar{c}_{klm}^u + (\bar{D}^u_v)_{ij}^{kl} \bar{c}_{kl}^v, \end{aligned} \quad (3.35d)$$

$$\begin{aligned} \partial_t d_i^v &\simeq (A^v_u)^{jk} \partial_i d_{jk}^u + (A^v_v)^j \partial_i d_j^v + (A^v_w) \partial_i w + (D^v_u)_i^k c_k^u + (D^v_u)_i^{kl} c_{kl}^u \\ &\quad + (D^v_v)_i^k c_k^v + (\bar{D}^v_u)_i^{kl} \bar{c}_{kl}^u + (\bar{D}^v_u)_i^{klm} \bar{c}_{klm}^u + (\bar{D}^v_v)_i^{kl} \bar{c}_{kl}^v, \end{aligned} \quad (3.35e)$$

$$\begin{aligned} \partial_t w &\simeq (A^w_u)^{ijk} \partial_i d_{jk}^u + (A^w_v)^{ij} \partial_i d_j^v + (A^w_w)^i \partial_i w + (D^w_u)^k c_k^u + (D^w_u)^{kl} c_{kl}^u \\ &\quad + (D^w_v)^k c_k^v + (\bar{D}^w_u)^{kl} \bar{c}_{kl}^u + (\bar{D}^w_u)^{klm} \bar{c}_{klm}^u + (\bar{D}^w_v)^{kl} \bar{c}_{kl}^v, \end{aligned} \quad (3.35f)$$

where the constant matrices (D^X_Y) and (\bar{D}^X_Y) ($X, Y = u, v, w$) are the reduction parameters.

Since the reduction parameters are contracted with auxiliary constraints and the symmetric part of the \bar{c} vanishes we assume without loss of generality that the \bar{D} symmetrized in the upper indices vanish:

$$\begin{aligned} (\bar{D}^X_Y)^{(kl)} &= 0, & (\bar{D}^X_Y)^{(klm)} &= 0, & (\bar{D}^X_Y)_i^{(kl)} &= 0, \\ (\bar{D}^X_Y)_i^{(klm)} &= 0, & (\bar{D}^X_Y)_{ij}^{(kl)} &= 0, & (\bar{D}^X_Y)_{ij}^{(klm)} &= 0. \end{aligned} \quad (3.36)$$

Moreover, since $d_{ij}^u = d_{(ij)}^u$ the reduction variables satisfy

$$\begin{aligned}
(D^u_u)^{kl} &= (D^u_u)^{(kl)}, & (D^u_u)_i{}^{kl} &= (D^u_u)_i{}^{(kl)}, & (D^v_u)^{kl} &= (D^v_u)^{(kl)}, \\
(D^v_u)_i{}^{kl} &= (D^v_u)_i{}^{(kl)}, & (D^w_u)^{kl} &= (D^w_u)^{(kl)}, & (\bar{D}^u_u)^{klm} &= (\bar{D}^u_u)^{k(lm)}, \\
(\bar{D}^u_u)_i{}^{klm} &= (\bar{D}^u_u)_i{}^{k(lm)}, & (\bar{D}^v_u)^{klm} &= (\bar{D}^v_u)^{k(lm)}, & (\bar{D}^v_u)_i{}^{klm} &= (\bar{D}^v_u)_i{}^{k(lm)}, \\
(\bar{D}^w_u)^{klm} &= (\bar{D}^w_u)^{k(lm)}, & (\bar{D}^u_u)_{ij}{}^{klm} &= (\bar{D}^u_u)_{(ij)}{}^{k(lm)}, & (\bar{D}^u_u)_{ij}{}^{kl} &= (\bar{D}^u_u)_{(ij)}{}^{kl}, \\
(D^u_u)_{ij}{}^{kl} &= (D^u_u)_{(ij)}{}^{(kl)}, & (D^u_u)_{ij}{}^k &= (D^u_u)_{(ij)}{}^k, \\
(D^u_v)_{ij}{}^k &= (D^u_v)_{(ij)}{}^k.
\end{aligned} \tag{3.37}$$

In a representation with the state vector

$$(u, d_i^u, v, d_{ij}^u, d_i^v, w) \tag{3.38}$$

the principal matrix of the system (3.35) is

$$\mathcal{A}_{1ij}^{p,kl} = \begin{pmatrix} (D^u_u)^p & (\tilde{D}^u_u)^{pk} & (D^u_v)^p & (\bar{D}^u_u)^{pkl} & (\bar{D}^u_v)^{pk} & 0 \\ (D^u_u)_i{}^p & (\tilde{D}^u_u)_i{}^{pk} & (D^u_v)_i{}^p & (\bar{D}^u_u)_i{}^{pkl} & (\bar{D}^u_v)_i{}^{pk} & 0 \\ (D^v_u)^p & (\tilde{D}^v_u)^{pk} & (D^v_v)^p & (\bar{D}^v_u)^{pkl} & (\bar{D}^v_v)^{pk} & 0 \\ (D^u_u)_{ij}{}^p & (\tilde{D}^u_u)_{ij}{}^{pk} & (D^u_v)_{ij}{}^p & (A^u_u)^{(k,l)}_{ij} \delta_i^p \delta_j^k + (\bar{D}^u_u)_{ij}{}^{pkl} & A^u_v \delta_i^p \delta_j^k + (\bar{D}^u_v)_{ij}{}^{pk} & 0 \\ (D^v_u)_i{}^p & (\tilde{D}^v_u)_i{}^{pk} & (D^v_v)_i{}^p & (A^v_u)^{kl} \delta_i^p \delta_j^k + (\bar{D}^v_u)_i{}^{pkl} & (A^v_v)^k \delta_i^p \delta_j^k + (\bar{D}^v_v)_i{}^{pk} & (A^v_w) \delta_i^p \\ (D^w_u)^p & (\tilde{D}^w_u)^{pk} & (D^w_v)^p & (A^w_u)^{pkl} + (\bar{D}^w_u)^{pkl} & (A^w_v)^{pk} + (\bar{D}^w_v)^{pk} & (A^w_w)^p \end{pmatrix}, \tag{3.39}$$

where $(\tilde{D}^X_Y)_{*}^{\star} := (\bar{D}^X_Y)_{*}^{\star} + (D^X_Y)_{*}^{\star}$.

3.6. Symmetric hyperbolicity

Definitions of symmetric hyperbolicity: Now we show that the following definitions of third order symmetric hyperbolicity are equivalent

Definition 3a. *The FT3S system (3.1) is called first order symmetric hyperbolic if there exists a first order reduction which is symmetric hyperbolic in the usual first order sense [14]. That is, there exists a choice of reduction parameters and a Hermitian matrix $H_1^{ij,kl} = H_1^{(ij)(kl)}$ which is positive definite in the space of symmetric tensors such that*

$$H_1^{ij,kl} \mathcal{A}_{1kl}^{p,mn} \tag{3.40}$$

is Hermitian for all p .

The matrix $H_1^{ij,kl}$ is symmetric in (i, j) and (k, l) , because we defined the reduction variable d_{ij}^u symmetric.

Definition 3b. *The FT3S system (3.1) is called FT3S symmetric hyperbolic if there exists a Hermitian matrix $H_3^{ij,kl} = H_3^{(ij)(kl)}$ which is positive definite in the space of symmetric tensors such that*

$$S_{ij} H_3^{ij,kl} \mathcal{A}_{3kl}^{p,mn} s_p s_{mn} \tag{3.41}$$

is Hermitian for every spatial vector s .

As before we denote a positive definite Hermitian matrix $H_3^{ij\,kl} = H_3^{(ij)\,(kl)}$ which makes (3.41) Hermitian a *symmetrizer*. If $H_3^{ij\,kl}$ makes (3.41) Hermitian, but is not necessarily positive definite then we call it a *candidate symmetrizer*. It is straightforward to check that given an FT3S symmetrizer $H_3^{ij\,kl}$ the energy

$$E = \int d^3x \epsilon = \int d^3x u_{ij}^\dagger H_3^{ij\,kl} u_{ij} \quad (3.42)$$

is conserved up to non principal terms, i.e. $\partial_t E \simeq 0$.

Def. 3a \Rightarrow Def. 3b: Given an FT3S system which satisfies definition 3a there exist, according to the usual definition of symmetric hyperbolicity for first order systems [14], reduction parameters, D and \bar{D} , and a matrix $H_1^{ij\,kl} = H_1^{(ij)\,(kl)}$ such that the product $H_1^{ij\,kl} \mathcal{A}_{1\,kl}^{p\,mn}$ is Hermitian for every p . Moreover the matrix $H_1^{ij\,kl}$ is positive definite in the space of symmetric tensors.

Using the state vector (3.38) both $H_1^{ij\,kl}$ and $\mathcal{A}_{1\,kl}^{p\,mn}$ are decomposed into 6×6 blocks. By grouping the variables as

$$(u, d_i^u, v \mid d_{ij}^u, d_i^v, w) \quad (3.43)$$

we identify four 3×3 sub matrices in H_1 and \mathcal{A}_1^p , where H_1 has the form

$$H_1^{ij\,kl} = \begin{pmatrix} H_{11}^{i\,k} & H_{12}^{i\,kl} \\ H_{21}^{ij\,k} & H_{22}^{ij\,kl} \end{pmatrix}. \quad (3.44)$$

We are now interested in the lower right 3×3 sub matrix of the product of H_1 and \mathcal{A}_1^p . It turns out that this sub matrix contains the FT3S conservation equation, i.e. the condition that (3.41) is Hermitian.

The lower right block of the product $H_1^{ij\,kl} \mathcal{A}_{1\,kl}^{p\,mn}$ is

$$\begin{aligned} H_{21}^{ij\,k} & \begin{pmatrix} (\bar{D}^u_u)^{pmn} & (\bar{D}^u_v)^{pm} & 0 \\ (\bar{D}^u_u)_k{}^{pmn} & (\bar{D}^u_v)_k{}^{pm} & 0 \\ (\bar{D}^v_u)^{pmn} & (\bar{D}^v_v)^{pm} & 0 \end{pmatrix} \\ & + H_{22}^{ij\,kl} \begin{pmatrix} (A^u_u)^{(m}\delta_{(k}^n)\delta_{l)}^p + (\bar{D}^u_u)_{kl}{}^{pmn} & A^u_v\delta_{(k}^p\delta_{l)}^m + (\bar{D}^u_v)_{kl}{}^{pm} & 0 \\ (A^v_u)^{mn}\delta_k^p + (\bar{D}^v_u)_k{}^{pmn} & (A^v_v)^m\delta_k^p + (\bar{D}^v_v)_k{}^{pm} & (A^v_w)\delta_k^p \\ (A^w_u)^{pmn} + (\bar{D}^w_u)^{pmn} & (A^w_v)^{pm} + (\bar{D}^w_v)^{pm} & (A^w_w)^p \end{pmatrix}. \end{aligned} \quad (3.45)$$

By assumption this matrix is Hermitian, because it is a quadratic subblock on the diagonal of the Hermitian matrix $H_1^{ij\,kl} \mathcal{A}_{1\,kl}^{p\,mn}$.

Furthermore, when we contract the index p in (3.45) with an arbitrary spatial vector s_p and the full matrix from the left and right with S_{ij} and $S_{mn} = \text{diag}(s_m s_n, s_m, 1)$ respectively then the result is still Hermitian, because S_{ij} and S_{mn} are Hermitian.

Using the fact that the symmetrization of the reduction parameters \bar{D} in all upper indices vanishes according to (3.36) it follows that all terms in (3.45) that

contain reduction parameters vanish after the contractions with s_p , S_{ij} and S_{mn} . The remaining terms are

$$S_{ij}H_{22}^{ij\,kl} \begin{pmatrix} (A^u_u)^{(k}\delta_{(m}^l)\delta_n^p & A^u_v\delta_{(k}^p\delta_l^m & 0 \\ (A^v_u)^{mn}\delta_k^p & (A^v_v)^m\delta_k^p & (A^v_w)\delta_k^p \\ (A^w_u)^{pmn} & (A^w_v)^{pm} & (A^w_w)^p \end{pmatrix} s_p S_{mn} = S_{ij}H_{22}^{ij\,kl} \mathcal{A}_{3kl}^{p\,mn} s_p S_{mn}.$$

It is clear that $H_{22}^{ij\,kl} = H_{22}^{(ij)(kl)}$, and since it is a principal minor of the positive definite $H_1^{ij\,kl}$ it is positive definite as well. With the identification

$$H_3^{ij\,kl} = H_{22}^{ij\,kl} \quad (3.46)$$

this shows that the FT3S system is symmetric hyperbolic in the sense of definition 3b.

Def. 3b \Rightarrow Def. 3a: Given a matrix $H_3^{ij\,kl} = H_3^{(ij)(kl)}$ which satisfies (3.41) we now construct a symmetric hyperbolic first order reduction of (3.1). At first it is convenient to make a partial choice of the reduction parameters such that the first three rows and columns of (3.39) vanish. This is achieved by choosing all reduction parameters $D^X_Y = 0$ ($X, Y = u, v, w$) and in addition

$$\begin{aligned} (\bar{D}^u_u)^{pk} &= 0, & (\bar{D}^u_u)_i{}^{pk} &= 0, & (\bar{D}^v_u)^{pk} &= 0, & (\bar{D}^u_u)_{ij}{}^{pk} &= 0, \\ (\bar{D}^v_u)_i{}^{pk} &= 0, & (\bar{D}^w_u)^{pk} &= 0, & (\bar{D}^u_u)^{pkl} &= 0, & (\bar{D}^u_u)_i{}^{pkl} &= 0, \\ (\bar{D}^v_u)^{pkl} &= 0, & (\bar{D}^u_v)^{pk} &= 0, & (\bar{D}^u_v)_i{}^{pk} &= 0, & (\bar{D}^v_v)^{pk} &= 0. \end{aligned} \quad (3.47)$$

The next step is to make the ansatz

$$H_1^{ij\,kl} = \begin{pmatrix} \Gamma^{ik} & 0 \\ 0 & H_3^{ij\,kl} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \gamma^{ik} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & H_3^{ij\,kl} \end{pmatrix}, \quad (3.48)$$

where the 2×2 decomposition here is to be understood in the same sense as in (3.44). Obviously this matrix is positive definite when $H_3^{ij\,kl}$ is. Hence, what needs to be shown with this ansatz is that the remaining reduction parameters can be chosen such that (3.40) is Hermitian for all p .

We define

$$J^{p\,ij\,mn} := H_3^{ij\,kl} \begin{pmatrix} (\bar{D}^u_u)_{kl}{}^{pmn} & (\bar{D}^u_v)_{kl}{}^{pm} & 0 \\ (\bar{D}^v_u)_k{}^{pmn} & (\bar{D}^v_v)_k{}^{pm} & 0 \\ (\bar{D}^w_u)^{pmn} & (\bar{D}^w_v)^{pm} & 0 \end{pmatrix} \quad (3.49)$$

and

$$T^{p\,ij\,nm} := H_3^{ij\,kl} \mathcal{A}_{3kl}^{p\,mn}. \quad (3.50)$$

Since the form of $\mathcal{A}_{1ij}^{p\,kl}$ and $H_1^{ij\,kl}$ has been simplified by taking the partial choice (3.47) and the ansatz (3.48) respectively we only need to show that there

exist reduction parameters such that the matrix

$$J^{p\,ij\,mn} + T^{p\,ij\,nm} \quad (3.51)$$

is Hermitian for all p . In this equation $T^{p\,ij\,nm}$ is fixed because we assume an FT3S system with given symmetrizer.

The condition that $H_3^{ij\,kl}$ is a candidate symmetrizer is equivalent to

$$T^{(p\,ij\,nm)} = T^{\dagger(p\,ij\,nm)}, \quad (3.52)$$

because for all tensors $X^{p\,ij\,kl}$ the equivalence

$$X^{(p\,ij\,kl)} = 0 \quad \Leftrightarrow \quad s_p s_i s_j X^{p\,ij\,kl} s_k s_l = 0 \quad \forall s \quad (3.53)$$

holds.

Now we need to find an appropriate $J^{p\,ij\,nm}$. In order to be able to solve (3.49) for the reduction parameters it needs to satisfy certain symmetries:

$$J^{(p\,ij\,kl)} = 0, \quad J^{p\,ij\,kl} = J^{p\,(ij)\,(kl)}. \quad (3.54)$$

Note that $J^{(p\,ij\,kl)} = 0$ implies that the last column of $J^{p\,ij\,kl}$ vanishes.

One can prove the existence of a $J^{p\,ij\,kl}$ which satisfies (3.54) and makes (3.51) Hermitian by construction. With the definition

$$V^{p\,ij\,kl} := T^{p\,ij\,kl} - T^{\dagger p\,kl\,ij} \quad (3.55)$$

the condition that (3.51) is Hermitian becomes

$$J^{p\,ij\,kl} - J^{\dagger p\,kl\,ij} = -V^{p\,ij\,kl}. \quad (3.56)$$

In appendix Appendix B we show that using the ansatz

$$J^{p\,ij\,kl} = \sum_{\pi \in S_5} x_{\pi} V^{\pi(p)\,\pi(i)\,\pi(j)\,\pi(k)\,\pi(l)} \quad (3.57)$$

the system (3.54),(3.56) becomes a linear system on the x_{π} , which can be solved if $V^{(p\,ij\,kl)} = 0$.

The latter condition is satisfied by assumption. Hence, multiplication of the resulting $J^{p\,ij\,kl}$ from the left by H_3^{-1} (which exists, because H_3 is positive definite) shows that there exists a first order reduction which is symmetric hyperbolic and has the symmetrizer (3.48). \square

4. Higher order systems

In the following sections we extend the notions of strong and symmetric hyperbolicity to a certain type of higher order in space systems. As Gundlach and Martín-García in [11] we do not consider the most general first order in time, N th order in space system, but rather the subset for which a first order reduction exists. Here we describe these systems and establish our notation.

4.1. FTNS systems

Notation: We start by describing the notation that we use to present FTNS systems efficiently. The equations of motion will be given for fields v^μ , where v^μ denotes a vector of fields which can appear at most $N - \mu$ times differentiated in the FTNS system. For reasons that will become clear later we also denote fields with that property *variables with μ implicit derivatives*. To denote derivatives acting on v^ν we define for $\mu = 0, \dots, N - 1$, $\nu = 0, \dots, \mu$ and $\rho = 1, \dots, \mu - \nu$ operators

$$\hat{A}^\mu_\nu := (A^\mu_\nu)^{i_1 \dots i_{\mu-\nu+1}} \partial_{i_1 \dots i_{\mu-\nu+1}}, \quad (4.1a)$$

$$\hat{B}^\mu_{\rho\nu} := (B^\mu_{\rho\nu})^{i_1 \dots i_{\mu-\nu-\rho+1}} \partial_{i_1 \dots i_{\mu-\nu-\rho+1}}, \quad (4.1b)$$

with constant matrices $(A^\mu_\nu)^{i_1 \dots i_{\mu-\nu+1}}$ and $(B^\mu_{\rho\nu})^{i_1 \dots i_{\mu-\nu-\rho+1}}$. Since the number of “derivative indices” (the indices denoted by lower case Latin letters) in these matrices is fixed through μ , ν and ρ we also use the abbreviations

$$(A^\mu_\nu)^{\underline{i}} := (A^\mu_\nu)^{i_1 \dots i_{\mu-\nu+1}}, \quad (4.2a)$$

$$(A^\mu_\nu)^{i_1 \dots i_\sigma \underline{j}} := (A^\mu_\nu)^{i_1 \dots i_\sigma j_1 \dots j_{\mu-\nu-\sigma+1}}, \quad (4.2b)$$

$$(A^\mu_\nu)^{i_1 \dots i_\sigma \underline{i}} := (A^\mu_\nu)^{i_1 \dots i_\sigma i_{\sigma+1} \dots i_{\mu-\nu+1}}, \quad (4.2c)$$

i.e. an underlined lower case Latin letter means “fill in an appropriate number of derivative indices”. Analog notations are used for the other objects that appear here. The fields v^μ may also appear undifferentiated, i.e. in the form $A^\mu_{\mu+1} v^{\mu+1}$ or $B^\mu_{(\mu-\nu+1)\nu} v^\nu$. For efficiency we use the same notation in that case:

$$\hat{A}^\mu_{\mu+1} v^{\mu+1} := A^\mu_{\mu+1} v^{\mu+1} =: (A^\mu_{\mu+1})^{\underline{i}} \partial_{\underline{i}} v^{\mu+1}, \quad (4.3a)$$

$$\hat{B}^\mu_{(\mu-\nu+1)\nu} v^\nu := B^\mu_{(\mu-\nu+1)\nu} v^\nu =: (B^\mu_{(\mu-\nu+1)\nu})^{\underline{i}} \partial_{\underline{i}} v^\nu. \quad (4.3b)$$

Evolution equations: We define an FTNS system as a system of equations of the form

$$\partial_t v^\mu = \sum_{\nu=0}^{\mu+1} \hat{A}^\mu_\nu v^\nu + \sum_{\nu=0}^{\mu} \sum_{\rho=1}^{\mu-\nu+1} \hat{B}^\mu_{\rho\nu} v^\nu + s^\mu \quad (4.4a)$$

$$\partial_t v^{N-1} = \sum_{\nu=0}^{N-1} \hat{A}^{N-1}_\nu v^\nu + \sum_{\nu=0}^{N-1} \sum_{\rho=1}^{N-\nu} \hat{B}^{N-1}_{\rho\nu} v^\nu + s^{N-1}, \quad (4.4b)$$

with $\mu = 0, \dots, N - 2$ and source terms s^μ , s^{N-1} (the source terms do not contain the v^μ). Note that FT2S systems are the first order in time, second order in space systems treated in [11] and FT1S systems are fully first order systems. If we consider the equation of motion for v^μ in (4.4) then the left hand side, $\partial_t v^\mu$, is a first order derivative and in the right hand side the highest derivative acting on v^ν has order $\mu - \nu + 1$. If we consider v^μ as a variable which contains μ derivatives implicitly then the counting of derivatives gives at both sides $\mu + 1$. Therefore it is helpful to think of the v^μ in that way, which explains our terminology.

Principal part: We will see that one can define strong and symmetric hyperbolicity of FTNS systems through the coefficients of the highest order derivatives in (4.4). Therefore we call

$$\partial_t v^\mu \simeq \sum_{\nu=0}^{\mu+1} \hat{A}^\mu_{\nu} v^\nu, \quad \partial_t v^{N-1} \simeq \sum_{\nu=0}^{N-1} \hat{A}^{N-1}_{\nu} v^\nu, \quad (4.5)$$

with $\mu = 0, \dots, N-2$ the *principal part of the FTNS system*. Furthermore we denote the matrix

$$\mathcal{A}_N^p \underline{i}^j = \left((\Delta_{\mu\nu}^N)^{p(j)}_{(\underline{i})\underline{k}} (\tilde{A}^\mu_{\nu})^{\underline{k}} \right)_{\mu=0, \dots, N-1}^{\nu=0, \dots, N-1}, \quad (4.6)$$

with (\underline{i}) meaning symmetrization and

$$(\tilde{A}^\mu_{\nu})^{\underline{i}} := \begin{cases} (A^\mu_{\nu})^{\underline{i}} & \text{for } \nu \leq \mu + 1 \\ 0 & \text{for } \nu > \mu + 1, \end{cases} \quad (4.7a)$$

$$(\Delta_{\mu\nu}^N)^j_{\underline{i}\underline{k}} := \begin{cases} \delta_{i_1}^{j_1} \dots \delta_{i_{N-\mu-1}}^{j_{N-\mu-1}} \delta_{k_1}^{j_{N-\mu}} \dots \delta_{k_{\mu-\nu+1}}^{j_{N-\nu}} & \text{for } \mu \leq N-2, \nu \leq \mu \\ \delta_{i_1}^{j_1} \dots \delta_{i_{N-\mu-1}}^{j_{N-\mu-1}} & \text{for } \mu \leq N-2, \nu = \mu + 1 \\ \delta_{k_1}^{j_1} \dots \delta_{k_{N-\nu}}^{j_{N-\nu}} & \text{for } \mu = N-1, \nu \leq N-1 \\ 0 & \text{for } \nu > \mu + 1 \end{cases} \quad (4.7b)$$

for $\mu, \nu \leq N-1$ the *principal matrix of the FTNS system*. In the variables $u_{\underline{i}} = (\partial_{i_1} \dots \partial_{i_{N-\mu-1}} v^\mu)_{\mu=0, \dots, N-1}$ the principal part of the FTNS system can be written as

$$\partial_t u_{\underline{i}} \simeq \mathcal{A}_N^p \underline{i}^j \partial_p u_{\underline{j}}. \quad (4.8)$$

Principal symbol: The *principal symbol* of the FTNS system (4.4) is

$$P_N^s = S^N \underline{i} \mathcal{A}_N^p \underline{j} s_p S_{\underline{j}}^N, \quad (4.9)$$

where

$$S_{\underline{j}}^N = \text{diag}(s_{j_1} \dots s_{j_{N-1}}, s_{j_1} \dots s_{j_{N-2}}, \dots, s_{j_1}, 1). \quad (4.10)$$

5. Higher order strong hyperbolicity

In this section we consider strong hyperbolicity of FTNS systems. In analogy to the case of FT3S systems we introduce an iterative reduction procedure, $\text{FTNS} \rightarrow \text{FT}(N-1)\text{S} \rightarrow \dots \rightarrow \text{FT1S}$ and use this to define strong hyperbolicity for FTNS systems without referring to the reduction.

5.1. Reduction to order $(N-1)$

Reduction variables and auxiliary constraints: We begin with the description of reductions to order $(N-1)$. The starting point is the FTNS system (4.4).

Using the same procedure that was described in detail for FT3S systems in section 3.2 we construct FT($N - 1$)S reductions of (4.4). We define the reduction variables

$$d_i := \partial_i v^0 \quad (5.1)$$

and derive from (4.4a) their equation of motion:

$$\partial_t d_i = (A^0_0)^j \partial_i \partial_j v^0 + A^0_1 \partial_i v^1 + \hat{B}^0_{10} \partial_i v^0 + \partial_i s^0. \quad (5.2)$$

The auxiliary constraints introduced with the new reduction variable are

$$c_i := \partial_i v^0 - d_i, \quad (5.3a)$$

$$c_{i_1 \dots i_\sigma} := \partial_{i_1} \dots \partial_{i_{\sigma-1}} d_{i_\sigma} - \partial_{(i_1} \dots \partial_{i_{\sigma-1}} d_{i_\sigma)}. \quad (5.3b)$$

One can show that for $\sigma > 2$ the constraints $c_{i_1 \dots i_\sigma}$ can be written as linear combinations of derivatives of the c_{ij} . The proof can be done through induction with the induction step

$$c_{i_1 \dots i_\sigma} = \frac{1}{\sigma} \sum_{\mu=1}^{\sigma-1} \partial_{i_\mu} c_{i_1 \dots i_{\rho-1} i_{\rho-2} \dots i_\sigma} + \frac{2}{\sigma(\sigma-1)} \sum_{\nu=1}^{\sigma-1} \partial_{i_1} \dots \partial_{i_{\nu-1}} \partial_{i_{\nu+1}} \dots \partial_{i_\sigma} c_{i_\mu i_\sigma}. \quad (5.4)$$

FT($N - 1$)S reduction: In analogy to section 3.2 we come to FT($N - 1$)S reductions by adding the constraints c_i and c_{ij} as well as their derivatives to (4.4) and (5.2). If we restrict to those constraint additions which appear in the resulting FT($N - 1$)S principal part then we get the following class of FT($N - 1$)S systems

$$\partial_t v^0 = (A^0_0)^k \partial_k v^0 + (A^0_1) v^1 + (B^0_{10}) v^0 + s^0 + (D^0)^k c_k + (\bar{D}^0)^{kj} c_{kj}, \quad (5.5a)$$

$$\partial_t d_i = (A^0_0)^j \partial_i \partial_j v^0 + A^0_1 \partial_i v^1 + B^0_{10} \partial_i v^0 + \partial_i s^0 + (D^0)_i^k c_k + (\bar{D}^0)_i^{kj} c_{kj}, \quad (5.5b)$$

$$\begin{aligned} \partial_t v^\mu &= (A^\mu_0)^{k_1 \dots k_{\mu+1}} \partial_{k_1} \dots \partial_{k_\mu} d_{k_{\mu+1}} + \sum_{\nu=1}^{\mu+1} \hat{A}^\mu_{\nu} v^\nu + \sum_{\nu=0}^{\mu} \sum_{\rho=1}^{\mu-\nu+1} \hat{B}^\mu_{\rho\nu} v^\nu + s^\mu \\ &\quad + (D^\mu)^{k_1 \dots k_\mu} \partial_{k_1} \dots \partial_{k_{\mu-1}} c_{k_\mu} + (\bar{D}^\mu)^{k_1 \dots k_{\mu+1}} \partial_{k_1} \dots \partial_{k_{\mu-1}} c_{k_\mu k_{\mu+1}}, \end{aligned} \quad (5.5c)$$

$$\begin{aligned} \partial_t v^{N-1} &= (A^{N-1}_0)^{k_1 \dots k_N} \partial_{k_1} \dots \partial_{k_{N-1}} d_{k_N} + s^{N-1} + \sum_{\nu=1}^{N-1} \hat{A}^{N-1}_{\nu} v^\nu \\ &\quad + \sum_{\nu=0}^{N-1} \sum_{\rho=1}^{N-\nu} \hat{B}^{N-1}_{\rho\nu} v^\nu + (D^{N-1})^{k_1 \dots k_{N-1}} \partial_{k_1} \dots \partial_{k_{N-2}} c_{k_{N-1}}, \\ &\quad + (\bar{D}^{N-1})^{k_1 \dots k_N} \partial_{k_1} \dots \partial_{k_{N-2}} c_{k_{N-1} k_N}, \end{aligned} \quad (5.5d)$$

where $\mu = 1, \dots, N - 2$ and the matrices denoted D and \bar{D} are the reduction parameters. Due to the antisymmetry of c_{ij} one can assume without loss of generality that the \bar{D} are antisymmetric in the last two indices. By applying this reduction procedure ($N - 1$) times we finally arrive at an FT1S system.

Auxiliary constraint evolution: By construction it is clear that there is a one-to-one correspondence between the solutions of (4.4) and the solutions of (5.5) which satisfy the auxiliary constraints (5.3). The reason is that the constraint evolution system is closed:

$$\partial_t c_i = ((A^0_0)^k + (D^0)^k) \partial_i c_k + D_i^k c_k + (\bar{D}^0)^{kj} \partial_i c_{kj} + \bar{D}_i^{kj} c_{kj}, \quad (5.6a)$$

$$\partial_t c_{ij} = D_{[j}^k \partial_{i]} c_k + \bar{D}_{[j}^{kl} \partial_{i]} c_{kl}. \quad (5.6b)$$

Having (5.4) and (5.6) one can show by induction that $\partial_t c_{i_1 \dots i_\sigma}$ is equal to a linear combination of the auxiliary constraints (5.3) and their spatial derivatives.

Principal part: The principal part of the FT($N-1$)S system (5.5) is

$$\partial_t v^0 \simeq ((A^0_0)^k + (D^0)^k) \partial_k v^0 + (\bar{D}^0)^{kj} \partial_k d_j, \quad (5.7a)$$

$$\partial_t d_i \simeq (B^0_{10} \delta_i^k + (D)_i^k) \partial_k v^0 + ((A^0_0)^j \delta_i^k + (\bar{D})_i^{kj}) \partial_k d_j + A^0_1 \partial_i v^1, \quad (5.7b)$$

$$\begin{aligned} \partial_t v^\mu &\simeq ((B^\mu_{10})^{\underline{k}} + (D^\mu)^{\underline{k}}) \partial_{k_1} \dots \partial_{k_\mu} v^0 \\ &\quad + ((A^\mu_0)^{\underline{k}} + (\bar{D}^\mu)^{\underline{k}}) \partial_{k_1} \dots \partial_{k_\mu} d_{k_{\mu+1}} + \sum_{\nu=1}^{\mu+1} \hat{A}^\mu_{\nu} v^\nu, \end{aligned} \quad (5.7c)$$

$$\begin{aligned} \partial_t v^{N-1} &\simeq ((B^{N-1}_{10})^{\underline{k}} + (D^{N-1})^{\underline{k}}) \partial_{k_1} \dots \partial_{k_{N-1}} v^0 \\ &\quad + ((A^{N-1}_0)^{\underline{k}} + (\bar{D}^{N-1})^{\underline{k}}) \partial_{k_1} \dots \partial_{k_{N-1}} d_{k_N} + \sum_{\nu=1}^{N-1} \hat{A}^{N-1}_{\nu} v^\nu, \end{aligned} \quad (5.7d)$$

where $\mu = 1, \dots, N-2$.

For the ordering of variables $(v^0, d_i, v^1, \dots, v^{N-1})$ the principal matrix of the FT($N-1$)S reduction (5.5) becomes

$$\mathcal{A}_{N-1}^{p, j}_{\underline{i}} = \begin{pmatrix} (\Delta_{00}^{N-1})_{\underline{i}\underline{k}}^{pj} ((A^0_0)^k + (D^0)^k) & (\Delta_{00}^{N-1})_{\underline{i}\underline{k}}^{pj} (\bar{D}^0)^{kj} & 0 \\ (\Delta_{00}^{N-1})_{\underline{i}\underline{k}}^{pj} ((B^0_{10}) \delta_i^k + (D)_i^k) & (\Delta_{00}^{N-1})_{\underline{i}\underline{k}}^{pj} ((A^0_0)^j \delta_i^k + (\bar{D})_i^{kj}) & (\tilde{\Delta}_{0(\nu-1)}^{N-1})_{\underline{i}\underline{k}}^{pj} (\tilde{A}^0_\nu) \delta_i^k \\ (\Delta_{(\mu-1)0}^{N-1})_{\underline{i}\underline{k}}^{pj} ((B^\mu_{10})^{\underline{k}} + (D^\mu)^{\underline{k}}) & (\Delta_{(\mu-1)0}^{N-1})_{\underline{i}\underline{k}}^{pj} ((A^\mu_0)^{\underline{k}j} + (\bar{D}^\mu)^{\underline{k}j}) & (\tilde{\Delta}_{(\mu-1)(\nu-1)}^{N-1})_{\underline{i}\underline{k}}^{pj} (\tilde{A}^\mu_\nu)^{\underline{k}} \end{pmatrix}, \quad (5.8)$$

where $\mu, \nu = 1, \dots, N-1$.

Note that

$$(\Delta_{00}^{N-1})_{\underline{i}\underline{k}}^{pj} (A^0_0)^{jN-1} \delta_{iN-1}^k = (\Delta_{00}^N)_{\underline{i}\underline{k}}^{pj} (A^0_0)^k, \quad (5.9a)$$

$$(\Delta_{00}^{N-1})_{\underline{i}\underline{k}}^{pj} (A^0_1) \delta_{iN-1}^k = (\Delta_{01}^N)_{\underline{i}}^{pj} (A^0_1), \quad (5.9b)$$

$$(\Delta_{(\mu-1)0}^{N-1})_{\underline{i}\underline{k}}^{pj} (A^\mu_0)^{\underline{k}jN-1} = (\Delta_{\mu 0}^N)_{\underline{i}\underline{k}}^{pj} (A^\mu_0)^{\underline{k}}, \quad (5.9c)$$

$$(\tilde{\Delta}_{(\mu-1)(\nu-1)}^{N-1})_{\underline{i}\underline{j}}^{\underline{k}} = (\tilde{\Delta}_{\mu\nu}^N)_{\underline{i}\underline{j}}^{\underline{k}}. \quad (5.9d)$$

Hence, if we rename $i \rightarrow i_{N-1}$ and $j \rightarrow j_{N-1}$ and assume vanishing reduction parameters \bar{D} then the FT($N-1$)S principal matrix has the FTNS principal matrix

as a submatrix:

$$\mathcal{A}_{N-1}^{p_{\underline{i}} j_{N-1}} = \begin{pmatrix} * & 0 \\ * & \mathcal{A}_N^{p_{\underline{i}} j} \end{pmatrix}. \quad (5.10)$$

The $\text{FT}(N-1)\text{S}$ principal symbol can be obtained by the appropriate contraction of the principal matrix with a spatial vector s :

$$P_{N-1}^s = \begin{pmatrix} ((A^0_0)^k + (D^0)^k) s_k & (\bar{D}^0)^{kj} s_k & 0 \\ ((B^0_{10})^k + (D)^k_i) s_k & ((A^0_0)^j + (\bar{D})^j_i) s_k & (\tilde{A}^0_\nu) s_i \\ ((B^\mu_{10})^{\underline{k}} + (D^\mu)^{\underline{k}}) s_{\underline{k}}^\mu & ((A^\mu_0)^{\underline{k}j} + (\bar{D}^\mu)^{\underline{k}j}) s_{\underline{k}}^\mu & (\tilde{A}^\mu_\nu)^{\underline{k}} s_{\underline{k}}^{\mu-\nu+1} \end{pmatrix}, \quad (5.11)$$

where $s_{\underline{k}}^\nu = s_{k_1} \dots s_{k_\nu}$.

5.2. *FTNS strong hyperbolicity*

Definitions of strong hyperbolicity: Having defined reductions of FTNS systems to $\text{FT}(N-1)\text{S}$ systems we now give two definitions of strong hyperbolicity for FTNS systems and show their equivalence. The first definition makes use of the $\text{FT}(N-1)\text{S}$ reduction.

Definition 4a. *The FTNS system (4.4) is called $\text{FT}(N-1)\text{S}$ strongly hyperbolic if there exists an $\text{FT}(N-1)\text{S}$ reduction (5.5) which is $\text{FT}(N-1)\text{S}$ strongly hyperbolic in the sense of definition 4b.*

The second definition does not rely on any reduction to lower order systems. Note that for $N = 1$ it is consistent with the standard definition of strong hyperbolicity for fully first order systems [14].

Definition 4b. *The FTNS system (4.4) is called FTNS strongly hyperbolic if there exist a constant $M_N > 0$ and a family of hermitian matrices $H_N(s)$ such that*

$$\begin{aligned} H_N(s) P_N^s &= (P_N^s)^\dagger H_N(s), \\ M_N^{-1} I &\leq H_N(s) \leq M_N I, \end{aligned} \quad (5.12)$$

where the matrix inequality is understood in the standard sense (2.11).

Equivalence of the definitions: We now demonstrate that the two definitions of strong hyperbolicity are equivalent. There is no major difference to the case of $N = 3$ which was discussed in section 3.3.

2+1 decomposition: For the proof we apply a 2+1 decomposition of the reduction variable d_i . Let q_a^A be the orthogonal projector of s , then the reduction variable is written as $d_i = q_i^A d_A + s_i d_s$, where $d_A s^A = 0$. With the state vec-

tor $(v^0, d_A, d_s, v^1, \dots, v^{N-1})$ the principal symbol (5.11) becomes

$$P_{N-1}^s A^B = \begin{pmatrix} ((A^0_0)^k + (D^0)^k) s_k & (\bar{D}^0)^{kj} s_k q_j^B & 0 & 0 \\ (D)_i^k s_k q_A^i & (\bar{D})_i^{kj} s_k q_j^B q_A^i & 0 & 0 \\ ((B^0_{10})^k \delta_i^k + (D)_i^k) s_k s^i & ((A^0_0)^j \delta_i^k + (\bar{D})_i^{kj}) s_k q_j^B s^i & (A^0_0)^j s_j & \tilde{A}^0_\nu \\ ((B^\mu_{10})^{\underline{k}} + (D^\mu)^{\underline{k}}) s_{\underline{k}}^\mu & ((A^\mu_0)^{\underline{k}j} + (\bar{D}^\mu)^{\underline{k}j}) s_{\underline{k}}^\mu q_j^B & (A^\mu_0)^{\underline{k}j} s_{\underline{k}}^\mu s_j & (\tilde{A}^\mu_\nu)^{\underline{k}} s_{\underline{k}}^{\mu-\nu+1} \end{pmatrix}. \quad (5.13)$$

Definition 4a \Rightarrow 4b: Assume that definition 4a is satisfied for an FT($N-1$)S reduction (5.5), i.e. there exist a constant $M_{N-1} > 0$ and a family of hermitian matrices $H_{N-1}(s)^{AB}$ such that

$$H_{N-1}(s)^{AB} P_{N-1}^s B^C = (P_{N-1}^s B^A)^\dagger H_{N-1}(s)^{BC}, \quad (5.14)$$

$$M_{N-1}^{-1} I^{AB} \leq H_{N-1}(s)^{AB} \leq M_{N-1} I^{AB},$$

where I^{AB} is the appropriate identity.

Since (5.13) is a block triangular matrix with the lower right diagonal block

$$\begin{pmatrix} (A^0_0)^j s_j & \tilde{A}^0_\nu \\ (A^\mu_0)^{\underline{k}} s_{\underline{k}}^\mu & (\tilde{A}^\mu_\nu)^{\underline{k}} s_{\underline{k}}^{\mu-\nu+1} \end{pmatrix} = P_N^s, \quad (5.15)$$

the same arguments used in section 3.3 can be applied to show that in an appropriate decomposition of $H_{N-1}(s)^{AB}$ the lower right block is a bounded symmetrizer of P_N^s . Hence, definition 4b is satisfied.

Definition 4b \Rightarrow 4a: Conversely, assuming definition 4b is satisfied for an FTNS system (4.4) one can identify an FT($N-1$)S reduction which is strongly hyperbolic. We make the partial choice of reduction parameters

$$(D^0)^k = -(A^0_0)^k, \quad (D)_i^k = -(B^0_{10}) \delta_i^k, \quad (D^\mu)^{\underline{k}} = -(B^\mu_{10})^{\underline{k}}, \quad (5.16)$$

for $\mu = 1, \dots, N-1$. With this choice (5.13) has the form

$$P_{N-1}^s A^B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & X_A^B & 0 \\ 0 & Y^B & P_N^s \end{pmatrix}, \quad (5.17)$$

where X_A^B is the same matrix, (3.26a), as in the FT3S case, $X_A^B = (\bar{D})_i^{kj} s_k q_j^B q_A^i$, and

$$Y^B = \begin{pmatrix} ((A^0_0)^j \delta_i^k + (\bar{D})_i^{kj}) s_k q_j^B s^i \\ ((A^\mu_0)^{\underline{k}j} + (\bar{D}^\mu)^{\underline{k}j}) s_{\underline{k}}^\mu q_j^B \end{pmatrix}. \quad (5.18)$$

The same procedure that we used for FT3S systems in section 3.3 allows the identification of a strongly hyperbolic FT($N-1$)S reduction. The key in this procedure is to choose

$$(\bar{D})_i^{kj} = i\lambda \varepsilon_i^{jk}, \quad (5.19)$$

where $\lambda \in \mathbb{R}$. With this the eigenvalues of X_A^B are $\pm\lambda$ and if λ is sufficiently large then one can show that definition 4a is satisfied using the assumption that the properties of the principal symbol in definition 4b hold for P_N^s . \square

6. Higher order symmetric hyperbolicity

In this section we show that one can extend to notion of symmetric hyperbolicity to higher order in space systems. For reasons discussed in section 3.4 we follow the strategy to employ a direct reduction to first order.

6.1. Reduction of FTNS systems to first order

Reduction variables: We start with the description of the reduction to first order for the FTNS system (4.4),

$$\partial_t v^\mu = \sum_{\nu=0}^{\mu+1} \hat{A}^\mu_{\nu} v^\nu + \sum_{\nu=0}^{\mu} \sum_{\rho=1}^{\mu-\nu+1} \hat{B}^\mu_{\rho\nu} v^\nu + s^\mu \quad (6.1a)$$

$$\begin{aligned} \partial_t v^{N-1} &= \sum_{\nu=0}^{N-1} \hat{A}^{N-1}_{\nu} v^\nu + \sum_{\nu=0}^{N-1} \sum_{\rho=1}^{N-\nu} \hat{B}^{N-1}_{\rho\nu} v^\nu \\ &\quad + s^{N-1}, \end{aligned} \quad (6.1b)$$

with $\mu = 0, \dots, N-2$. The reduction variables which we define are denoted d_ν^μ . The two indices have the following meaning:

μ	the reduction variable refers to v^μ in the original FTNS system
ν	the reduction variable has ν derivative indices ($1 \leq \nu \leq N - \mu - 1$)

The reduction variables are defined as

$$(d_1^\mu)_i = (d_1^\mu)_{\underline{i}} := \partial_i v^\mu, \quad (d_\nu^\mu)_{i_1 \dots i_\nu} = (d_\nu^\mu)_{\underline{i}} := \partial_{(i_1} (d_{\nu-1}^\mu)_{i_2 \dots i_\nu)}, \quad (6.2)$$

where $\mu = 0, \dots, N-2$, $\nu = 2, \dots, N - \mu - 1$. For convenience we also use the notation

$$(d_0^\mu) = (d_0^\mu)_{\underline{i}} := v^\mu. \quad (6.3)$$

One finds that the important variables for the principal part of the first order reduction are those with the highest number of derivative indices, i.e. $d_{N-\mu-1}^\mu$. We abbreviate them as

$$(d^\mu)_{\underline{i}} := (d_{N-\mu-1}^\mu)_{\underline{i}}, \quad (6.4)$$

where $\mu = 0, \dots, N-2$.

Unmodified equations of motion: Using these definitions the equations of motion for the reduction variables can be derived by taking derivatives of (6.1a). One finds

$$\begin{aligned} \partial_t(d_\nu^\mu)_{\underline{i}} &= \sum_{\rho=0}^{\mu+1} \hat{A}^\mu_{\rho} \partial_{i_1} \dots \partial_{i_\nu} v^\rho \\ &+ \sum_{\rho=0}^{\mu} \sum_{\sigma=1}^{\mu-\rho+1} \hat{B}^\mu_{\sigma\rho} \partial_{i_1} \dots \partial_{i_\nu} v^\rho + \partial_{i_1} \dots \partial_{i_\nu} s^\mu, \end{aligned} \quad (6.5a)$$

where $\mu = 0, \dots, N-2$, $\nu = 1, \dots, N-\mu-1$ and we used

$$\hat{A}^\mu_{\rho} \partial_{i_1} \dots \partial_{i_\nu} v^\rho := (A^\mu_{\rho})^{j_1 \dots j_{\mu-\rho+1}} \partial_{i_1} \dots \partial_{i_\nu} \partial_{j_1} \dots \partial_{j_{\mu-\rho+1}} v^\rho, \quad (6.6a)$$

$$\hat{B}^\mu_{\sigma\rho} \partial_{i_1} \dots \partial_{i_\nu} v^\rho := (B^\mu_{\sigma\rho})^{j_1 \dots j_{\mu-\rho-\sigma+1}} \partial_{i_1} \dots \partial_{i_\nu} \partial_{j_1} \dots \partial_{j_{\mu-\rho-\sigma+1}} v^\rho. \quad (6.6b)$$

The terms $\partial_{i_1} \dots \partial_{i_\nu} s^\mu$ in (6.5) do not contain the d_ν^μ or v^μ and can be seen as given source terms.

Auxiliary constraints: The reduction variables are subject to the following first order auxiliary constraints

$$(c_\nu^\mu)_{i_1 \dots i_\nu} = (c_\nu^\mu)_{\underline{i}} := \partial_{(i_1} (d_{\nu-1}^\mu)_{i_2 \dots i_\nu)} - (d_\nu^\mu)_{i_1 \dots i_\nu}, \quad (6.7a)$$

$$(\bar{c}_\nu^\mu)_{i_1 \dots i_{\nu+1}} = (\bar{c}_\nu^\mu)_{\underline{i}} := \partial_{i_1} (d_\nu^\mu)_{i_2 \dots i_{\nu+1}} - \partial_{(i_1} (d_\nu^\mu)_{i_2 \dots i_{\nu+1})}, \quad (6.7b)$$

where $\mu = 0, \dots, N-2$, $\nu = 1, \dots, N-\mu-1$.

First order reduction: As before we ask now, which first order systems can be constructed by adding the constraints (6.7) and their derivatives to the right hand sides of (6.1) and (6.5).

We note that lower order derivatives of the v^μ (i.e. derivatives of order $N-\mu-1$ or smaller) can be written as linear combinations of the constraints, their derivatives and undifferentiated reduction variables. For $\mu = 0, \dots, N-2$ and $\nu = 1, \dots, N-\mu-1$ one finds

$$\partial_{i_1} \dots \partial_{i_\nu} v^\mu = (d_\nu^\mu)_{\underline{i}} + \sum_{\rho=0}^{\nu-1} \partial_{i_1} \dots \partial_{i_\rho} (c_{\nu-\rho}^\mu)_{i_{\rho+1} \dots i_\nu} + \sum_{\rho=0}^{\nu-2} \partial_{i_1} \dots \partial_{i_\rho} (\bar{c}_{\nu-\rho-1}^\mu)_{i_{\rho+1} \dots i_\nu}, \quad (6.8)$$

where the sums are understood to vanish when the upper bound is smaller than the lower bound and the terms with $\rho = 0$ should be interpreted as the undifferentiated constraints.

One can prove this by induction over ν . For $\nu = 1$ we get

$$\partial_{i_1} v^\mu = (c_1^\mu)_{i_1} + (d_1^\mu)_{i_1}, \quad (6.9)$$

which is of the form (6.8). Assuming that (6.8) holds for a certain ν we get

$$\begin{aligned} \partial_{i_1} \dots \partial_{i_{\nu+1}} v^\mu &= \partial_{i_1} (d_\nu^\mu)_{i_2 \dots i_{\nu+1}} + \partial_{i_1} \sum_{\rho=1}^{\nu} \partial_{i_2} \dots \partial_{i_\rho} (c_{\nu-\rho+1}^\mu)_{i_{\rho+1} \dots i_{\nu+1}} \\ &\quad + \partial_{i_1} \sum_{\rho=1}^{\nu-1} \partial_{i_2} \dots \partial_{i_\rho} (\bar{c}_{\nu-\rho}^\mu)_{i_{\rho+1} \dots i_{\nu+1}}. \end{aligned} \quad (6.10)$$

In case $\nu < N - \mu - 1$ the first term on the right hand side can be rewritten:

$$\partial_{i_1} (d_\nu^\mu)_{i_2 \dots i_{\nu+1}} = \partial_{(i_1} (d_\nu^\mu)_{i_2 \dots i_{\nu+1})} + (\bar{c}_\nu^\mu)_{i_1 \dots i_{\nu+1}} = (d_{\nu+1}^\mu)_{\underline{i}} + (c_{\nu+1}^\mu)_{\underline{i}} + (\bar{c}_\nu^\mu)_{\underline{i}}. \quad (6.11)$$

Hence, defining $\tilde{\nu} = \nu + 1$ one gets

$$\partial_{i_1} \dots \partial_{i_{\tilde{\nu}}} v^\mu = (d_{\tilde{\nu}}^\mu)_{\underline{i}} + \sum_{\rho=0}^{\tilde{\nu}-1} \partial_{i_1} \dots \partial_{i_\rho} (c_{\tilde{\nu}-\rho}^\mu)_{i_{\rho+1} \dots i_{\tilde{\nu}}} + \sum_{\rho=0}^{\tilde{\nu}-2} \partial_{i_1} \dots \partial_{i_\rho} (\bar{c}_{\tilde{\nu}-\rho-1}^\mu)_{i_{\rho+1} \dots i_{\tilde{\nu}}} \quad (6.12)$$

for $\mu = 0, \dots, N-2$ and $\tilde{\nu} = 1, \dots, N-\mu-1$. Likewise one finds for $\mu = 0, \dots, N-2$ and $\nu = N - \mu$

$$\begin{aligned} \partial_{i_1} \dots \partial_{i_{N-\mu}} v^\mu &= \partial_{i_1} (d^\mu)_{i_2 \dots i_{N-\mu}} + \sum_{\rho=1}^{N-\mu-1} \partial_{i_1} \dots \partial_{i_\rho} (c_{N-\mu-\rho}^\mu)_{i_{\rho+1} \dots i_{N-\mu}} \\ &\quad + \sum_{\rho=1}^{N-\mu-2} \partial_{i_1} \dots \partial_{i_\rho} (\bar{c}_{N-\mu-\rho-1}^\mu)_{i_{\rho+1} \dots i_{N-\mu}}, \end{aligned} \quad (6.13)$$

which is just the derivative of (6.8) with $\nu = N - \mu - 1$. This shows that when deriving a first order reduction all lower order derivatives of the v^μ can be completely absorbed into the constraint additions and that up to constraint additions the highest order derivative of v^μ becomes a first order symmetrized derivative of d^μ .

Reduction parameters: The ambiguity of adding arbitrary linear combinations of the auxiliary constraints (6.7) to the right hand sides of the first order system is parametrized by using *reduction parameters*. We denote the constraint additions as

$$D^X \sigma_\nu c_\sigma^\nu := (D^X \sigma_\nu)^{i_1 \dots i_\sigma} (c_\sigma^\nu)_{i_1 \dots i_\sigma}, \quad \bar{D}^X \sigma_\nu \bar{c}_\sigma^\nu := (\bar{D}^X \sigma_\nu)^{i_1 \dots i_{\sigma+1}} (\bar{c}_\sigma^\nu)_{i_1 \dots i_{\sigma+1}}, \quad (6.14)$$

where $\nu = 0, \dots, N-2$ and $\sigma = 1, \dots, N-\nu-1$. Depending on the equation where we add those constraints the index X is either a single index (in the case of constraint additions to the right hand sides of v^μ) or an index-tuple $(\mu, \lambda, i_1, \dots, i_\lambda)$ (in the right hand sides of d_λ^μ). The matrices $(D^X \sigma_\nu)^{i_1 \dots i_\sigma}$ and $(\bar{D}^X \sigma_\nu)^{i_1 \dots i_{\sigma+1}}$ are the reduction parameters. Without loss of generality we assume the symmetry properties

$$\begin{aligned} (D^X \sigma_\nu)^{i_1 \dots i_\sigma} &= (D^X \sigma_\nu)^{(i_1 \dots i_\sigma)}, & (\bar{D}^X \sigma_\nu)^{(i_1 \dots i_{\sigma+1})} &= 0, \\ (\bar{D}^X \sigma_\nu)^{i_1 i_2 \dots i_{\sigma+1}} &= (\bar{D}^X \sigma_\nu)^{i_1 (i_2 \dots i_{\sigma+1})}. \end{aligned} \quad (6.15)$$

The constraint additions on the different equations are independent of each other. We use the short notation

$$C^X = \sum_{\nu=0}^{N-2} \sum_{\sigma=1}^{N-\nu-1} D^X \sigma_{\nu} c_{\sigma}^{\nu} + \sum_{\nu=0}^{N-2} \sum_{\sigma=1}^{N-\nu-1} \bar{D}^X \sigma_{\nu} \bar{c}_{\sigma}^{\nu}, \quad (6.16)$$

where X has the same meaning as in (6.14).

Reduced equations of motion: With these findings the right hand sides for the v^{μ} in the first order reductions of (6.1) have the form

$$\begin{aligned} \partial_t v^{\mu} &= C^{\mu} + s^{\mu} + \sum_{\nu=0}^{\mu+1} (A^{\mu}_{\nu})^{\underline{j}} (d^{\nu}_{\mu-\nu+1})_{\underline{j}} + \sum_{\nu=0}^{\mu} \sum_{\rho=1}^{\mu-\nu+1} (B^{\mu}_{\rho\nu})^{\underline{j}} (d^{\nu}_{\mu-\nu-\rho+1})_{\underline{j}}, \\ \partial_t v^{N-2} &= C^{N-2} + s^{N-2} + \sum_{\nu=0}^{N-2} (A^{N-2}_{\nu})^{\underline{j}} (d^{\nu})_{\underline{j}} + (A^{N-2}_{N-1}) v^{N-1} \\ &\quad + \sum_{\nu=0}^{N-2} \sum_{\rho=1}^{N-\nu-1} (B^{\mu}_{\rho\nu})^{\underline{j}} (d^{\nu}_{\mu-\nu-\rho+1})_{\underline{j}}, \\ \partial_t v^{N-1} &= \sum_{\nu=0}^{N-2} (A^{N-1}_{\nu})^{j\underline{i}} \partial_j (d^{\nu})_{\underline{i}} + (A^{N-1}_{N-1})^j \partial_j v^{N-1} + C^{N-1} + s^{N-1}, \end{aligned} \quad (6.17)$$

for $\mu = 0, \dots, N-3$. Likewise one finds the equations of motion for the reduction variables in the first order reduction

$$\begin{aligned} \partial_t (d^{\mu}_{\sigma})_{\underline{i}} &= (C^{\mu}_{\sigma})_{\underline{i}} + \partial_{i_1 \dots i_{\sigma}}^{\sigma} s^{\mu} + \sum_{\nu=0}^{\mu+1} (A^{\mu}_{\nu})^{\underline{j}} (d^{\nu}_{\mu+\sigma-\nu+1})_{i_1 \dots i_{\sigma} \underline{j}} \\ &\quad + \sum_{\nu=0}^{\mu} \sum_{\rho=1}^{\mu-\nu+1} (B^{\mu}_{\rho\nu})^{\underline{j}} (d^{\nu}_{\mu+\sigma-\nu-\rho+1})_{i_1 \dots i_{\sigma} \underline{j}}, \end{aligned} \quad (6.18a)$$

$$\begin{aligned} \partial_t (d^{\mu})_{\underline{i}} &= \sum_{\nu=0}^{\mu+1} (A^{\mu}_{\nu})^{\underline{j}} (\Delta^N_{\mu\nu})^{\underline{k}}_{\underline{i}\underline{j}} \partial_{(p} (d^{\nu})_{\underline{k})} + (C^{\mu}_{N-\mu-1})_{\underline{i}} + \partial_{i_1 \dots i_{N-\mu-1}}^{N-\mu-1} s^{\mu} \\ &\quad + \sum_{\nu=0}^{\mu} \sum_{\rho=1}^{\mu-\nu+1} (B^{\mu}_{\rho\nu})^{\underline{j}} (d^{\nu}_{N-\nu-\rho})_{i_1 \dots i_{N-\mu-1} \underline{j}}, \end{aligned} \quad (6.18b)$$

$$\begin{aligned} \partial_t (d^{N-2})_i &= \sum_{\nu=0}^{N-2} (A^{N-2}_{\nu})^{\underline{j}} \partial_{(i} (d^{\nu})_{\underline{j})} + A^{N-2}_{N-1} \partial_i v^{N-1} + (C^{N-2}_1)_i + \partial_i s^{N-2} \\ &\quad + \sum_{\nu=0}^{N-2} \sum_{\rho=1}^{N-\nu-1} (B^{N-2}_{\rho\nu})^{\underline{j}} (d^{\nu}_{N-\nu-\rho})_{i \underline{j}}, \end{aligned} \quad (6.18c)$$

where $\mu = 0, \dots, N-3$ and $\sigma = 1, \dots, N-\mu-2$. The C^{μ}_{σ} can be read off from (6.16), and in (6.18b) we used the symbol $(\Delta^N_{\mu\nu})^{\underline{k}}_{\underline{i}\underline{j}}$ which is defined in (4.7b). We call a system of the form (6.17), (6.18) a *first order reduction* or *FT1S reduction* of the FTNS system (6.1).

Principal part: We now write the principal part of the first order reduction (6.17),(6.18) in a standard form. The terms that contain derivatives in the constraint additions are

$$\begin{aligned}
C^X &\simeq \sum_{\nu=0}^{N-2} \sum_{\sigma=0}^{N-\nu-2} (D^X{}^{(\sigma+1)}{}_{\nu})^{i_1 \dots i_{\sigma+1}} \partial_{i_1} (d_{\sigma}^{\nu})_{i_2 \dots i_{\sigma+1}} \\
&\quad + \sum_{\nu=0}^{N-2} \sum_{\sigma=1}^{N-\nu-1} (\bar{D}^X{}^{\sigma}{}_{\nu})^{i_1 \dots i_{\sigma+1}} \partial_{i_1} (d_{\sigma}^{\nu})_{i_2 \dots i_{\sigma+1}} \\
&= \sum_{\nu=0}^{N-2} (D^X{}^1{}_{\nu})^{i_1} \partial_{i_1} v^{\nu} + \sum_{\nu=0}^{N-3} \sum_{\sigma=1}^{N-\nu-2} (\bar{D}^X{}^{\sigma}{}_{\nu})^{i_1 \dots i_{\sigma+1}} \partial_{i_1} (d_{\sigma}^{\nu})_{i_2 \dots i_{\sigma+1}} \\
&\quad + \sum_{\nu=0}^{N-2} (\bar{D}^X{}^{(N-\nu-1)}{}_{\nu})^{i_1 \dots i_{N-\nu}} \partial_{i_1} (d_{\sigma}^{\nu})_{i_2 \dots i_{N-\nu}}, \tag{6.19}
\end{aligned}$$

where

$$(\tilde{D}^X{}^{\sigma}{}_{\nu})^{i_1 \dots i_{\sigma+1}} = (\tilde{D}^X{}^{\sigma}{}_{\nu})^{\dot{i}} := (D^X{}^{(\sigma+1)}{}_{\nu})^{i_1 \dots i_{\sigma+1}} + (\bar{D}^X{}^{\sigma}{}_{\nu})^{i_1 \dots i_{\sigma+1}} \tag{6.20}$$

and we used the symmetry properties (6.15) of the reduction parameters. The symbol \simeq means equality up to terms without derivatives and X has the same meaning as in (6.14). We write the state vector as

$$u_{\underline{i}} := ((d_{\sigma}^{\tilde{\mu}})_{\underline{i}}, v^{\mu}, (d^{\mu})_{\underline{i}}, w)^{\dagger}, \tag{6.21}$$

where the bounds for the indices are $\mu = 0, \dots, N-2$, $\tilde{\mu} = 0, \dots, N-3$ and $\sigma = 1, \dots, N-\tilde{\mu}-2$. The principal part of the system (6.17),(6.18) is then

$$\partial_t u_{\underline{i}} \simeq \mathcal{A}_1^p{}_{\underline{i}}{}^{\dot{j}} \partial_p u_{\underline{j}}, \tag{6.22}$$

where

$$\mathcal{A}_1^p{}_{\underline{i}}{}^{\dot{j}} = \begin{pmatrix} (\tilde{D}^{\tilde{\mu}}{}_{\sigma}{}^{\rho}{}_{\tilde{\nu}})_{\underline{i}}{}^{p\dot{j}} & (D^{\tilde{\mu}}{}_{\sigma}{}^1{}_{\nu})_{\underline{i}}{}^p & (\bar{D}^{\tilde{\mu}}{}_{\sigma}{}^{N-\nu-1}{}_{\nu})_{\underline{i}}{}^{p\dot{j}} & 0 \\ (\tilde{D}^{\mu}{}_{\rho}{}_{\tilde{\nu}})_{\underline{i}}{}^{p\dot{j}} & (D^{\mu}{}^1{}_{\nu})^p & (\bar{D}^{\mu}{}^{(N-\nu-1)}{}_{\nu})_{\underline{i}}{}^{p\dot{j}} & 0 \\ (\tilde{D}^{\mu}{}_{N-\mu-1}{}^{\rho}{}_{\tilde{\nu}})_{\underline{i}}{}^{p\dot{j}} & (D^{\mu}{}_{N-\mu-1}{}^1{}_{\nu})_{\underline{i}}{}^p & (\tilde{A}^{\mu}{}_{\nu})^k (\tilde{\Delta}_{\mu\nu}^N)_{ik}^{(p\dot{j})} + (\bar{D}^{\mu}{}_{N-\mu-1}{}^{N-\nu-1}{}_{\nu})_{\underline{i}}{}^{p\dot{j}} & 0 \\ (\tilde{D}^{(N-1)}{}_{\rho}{}_{\tilde{\nu}})_{\underline{i}}{}^{p\dot{j}} & (\tilde{D}^{(N-1)}{}^1{}_{\nu})^p & (\tilde{A}^{N-1}{}_{\nu})_{\underline{i}}{}^{p\dot{j}} + (\bar{D}^{(N-1)}{}^{(N-\nu-1)}{}_{\nu})_{\underline{i}}{}^{p\dot{j}} & (A^{N-1}{}_{N-1})^p \end{pmatrix} \tag{6.23}$$

and we used definition (4.7) for the symbols $\tilde{A}_{\mu\nu}$ and $\tilde{\Delta}_{\mu\nu}^N$. The range of the various indices in this expression is $\mu, \nu = 0, \dots, N-2$, $\tilde{\mu}, \tilde{\nu} = 0, \dots, N-3$, $\sigma = 1, \dots, N-\tilde{\mu}-2$ and $\rho = 1, \dots, N-\tilde{\nu}-2$.

Auxiliary constraint evolution: Having defined what we mean by first order reductions of the FTNS system (6.1) we note that again there is a one-to-one correspondence between the solutions of the first order reduction (6.17),(6.18) which satisfy the auxiliary constraints (6.7) and the solutions of the original FTNS system (6.1). This property of the reduced systems is a consequence of the construction

procedure, which leads to a closed constraint evolution system. To see that the constraint evolution system is closed is straightforward. One just uses equation (6.8) to express the reduction variables by derivatives of the v^μ and constraints. In the right hand sides of the constraint evolution system the derivatives of the v^μ cancel due to their symmetry in the derivative indices. This leads to the closed constraint evolution system. However, one obtains very lengthy expressions, which we present in appendix Appendix D.

6.2. FTNS symmetric hyperbolicity

Definitions of symmetric hyperbolicity: To get definitions of symmetric hyperbolicity for FTNS systems we generalize the second order definitions given in [11]. We start by defining candidate symmetrizers.

Definition 5a. *Given an FTNS system (6.1) we call a Hermitian matrix $H_N^{ij} = H_N^{(i)(j)}$ such that the product matrix*

$$S_{\underline{i}}^N H_N^{ik} \mathcal{A}_{N\underline{k}}^p \mathcal{J}_{s_p} S_{\underline{j}}^N, \quad (6.24)$$

is Hermitian for every s an FTNS candidate symmetrizer.

When we refer to lower order systems then we require the existence of a first order reduction such that there is a candidate symmetrizer in the usual first order sense:

Definition 5b. *We call a Hermitian matrix $H_1^{ij} = H_1^{(i)(j)}$ a first order candidate symmetrizer of (6.1) if there exists a first order reduction (6.17),(6.18) such that the product*

$$H_1^{ik} \mathcal{A}_{1\underline{k}}^p \mathcal{J}_{s_p}, \quad (6.25)$$

is Hermitian for every s .

In both cases we call a positive definite candidate symmetrizer a *symmetrizer*. With this it is straightforward to define symmetric hyperbolicity with and without reference to a first order reduction

Definition 6a. *The FTNS system (6.1) is called FTNS symmetric hyperbolic if there exists a positive definite FTNS candidate symmetrizer.*

Definition 6b. *The FTNS system (6.1) is called first order symmetric hyperbolic if there exists a positive definite first order candidate symmetrizer.*

Relationship between the definitions: Now we show for arbitrary N that definition 6b implies 6a. The proof of the reverse direction for arbitrary N involves very complicated expressions. We show in appendix Appendix B that for $N = 3$ it is indeed true that 6a implies 6b. For $N \leq 4$ we checked this using computer algebra. However, whether the statement holds for arbitrary N is an open question.

Construction of N th order from first order candidates: Let $H_1^{i,j}$ be the candidate symmetrizer of a first order reduction with principal matrix $\mathcal{A}_{1\bar{k}}^p{}^j$. We group the state vector (6.21) as

$$u_{\underline{i}} := ((d_{\sigma}^{\bar{\mu}})_{\underline{i}}, v^{\mu} \mid (d^{\mu})_{\underline{i}}, w)^{\dagger} \quad (6.26)$$

and in this way decompose H_1 and \mathcal{A}_1 consistently into

$$H_1^{i,j} = \begin{pmatrix} H_{11}^{i,j} & H_{12}^{i,j} \\ H_{21}^{i,j} & H_{22}^{i,j} \end{pmatrix}, \quad \mathcal{A}_{1\bar{k}}^p{}^j = \begin{pmatrix} \mathcal{A}_{11\bar{k}}^p{}^j & \mathcal{A}_{12\bar{k}}^p{}^j \\ \mathcal{A}_{21\bar{k}}^p{}^j & \mathcal{A}_{22\bar{k}}^p{}^j \end{pmatrix}, \quad (6.27)$$

where

$$\begin{aligned} \mathcal{A}_{12\bar{k}}^p{}^j &= \begin{pmatrix} (\bar{D}^{\bar{\mu}}{}_{\sigma}{}^{N-\nu-1}{}_{\nu})_{\underline{i}}{}^{pj} & 0 \\ (\bar{D}^{\mu}{}_{(N-\nu-1)}{}_{\nu})^{pj} & 0 \end{pmatrix}, \\ \mathcal{A}_{22\bar{k}}^p{}^j &= \begin{pmatrix} (\tilde{A}^{\mu}{}_{\nu})^{\bar{k}}(\tilde{\Delta}^N_{\mu\nu})_{(\underline{i})\bar{k}}^{p(j)} + (\bar{D}^{\mu}{}_{N-\mu-1}{}^{N-\nu-1}{}_{\nu})_{\underline{i}}{}^{pj} & 0 \\ (\tilde{A}^{N-1}{}_{\nu})^{pj} + (\bar{D}^{(N-1)}{}_{(N-\nu-1)}{}_{\nu})^{pj} & (A^{N-1}{}_{N-1})^p \end{pmatrix}, \end{aligned} \quad (6.28)$$

i.e. such that \mathcal{A}_{22} is the lower right 2×2 block of (6.23). In this decomposition the lower right block of the product $H_1^{i\bar{k}}\mathcal{A}_{1\bar{k}}^p{}^j$ is

$$H_{21}^{i\bar{k}}\mathcal{A}_{12\bar{k}}^p{}^j + H_{22}^{i\bar{k}}\mathcal{A}_{22\bar{k}}^p{}^j. \quad (6.29)$$

Hence, the matrix (6.29) is Hermitian for every p , because it is a principal minor of $H_1^{i\bar{k}}\mathcal{A}_{1\bar{k}}^p{}^j$.

Moreover, because $S_{\underline{i}}^N$ is Hermitian for every s , we get that (6.29) contracted from left and right with $S_{\underline{i}}^N$ is Hermitian for every p as well. Thus,

$$S_{\underline{i}}^N H_{21}^{i\bar{k}}\mathcal{A}_{12\bar{k}}^p{}^j S_p^N S_{\underline{j}}^N + S_{\underline{i}}^N H_{22}^{i\bar{k}}\mathcal{A}_{22\bar{k}}^p{}^j S_p^N S_{\underline{j}}^N \quad (6.30)$$

is Hermitian for every s . On the other hand

$$\mathcal{A}_{12\bar{k}}^p{}^j S_p^N S_{\underline{j}}^N = 0, \quad \mathcal{A}_{22\bar{k}}^p{}^j S_p^N S_{\underline{j}}^N = \mathcal{A}_{N\bar{k}}^p{}^j S_p^N S_{\underline{j}}^N, \quad (6.31)$$

because the symmetric part of the reduction parameters contained in \mathcal{A}_{12} and \mathcal{A}_{22} vanishes.

Since H_{22} is on the diagonal of H_1 it is Hermitian as well. Thus, with the identification

$$H_N^{i\bar{k}} = H_{22}^{i\bar{k}} \quad (6.32)$$

there exists an N th order candidate symmetrizer.

Positivity of the FTNS candidate symmetrizer: Moreover, if $H_1^{i,j}$ is positive definite then also $H_{22}^{i\bar{k}}$ is positive definite, because it is a principal minor. Hence, if there exists a first order reduction of (6.1) which is symmetric hyperbolic then (6.1) is also FTNS symmetric hyperbolic with the symmetrizer $H_N^{i\bar{k}} = H_{22}^{i\bar{k}}$.

Construction of a symmetric hyperbolic first order reduction: Now, for the reverse direction we assume a given FTNS symmetrizer, $H_N^{i\bar{j}}$, and would like to show that there exists a first order reduction with symmetrizer

$$H_1^{i\bar{j}} = \left(\begin{array}{ccc|c} \Gamma_\rho^{(i_1 \dots i_\rho)(j_1 \dots j_\rho)} & \dots & 0 & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 1 & 0 \\ \hline 0 & & & H_N^{i\bar{j}} \end{array} \right). \quad (6.33)$$

(in the 2×2 decomposition (6.27)) with

$$\Gamma_\rho^{i_1 \dots i_\rho j_1 \dots j_\rho} = \gamma^{i_1 j_1} \dots \gamma^{i_\rho j_\rho} \quad (6.34)$$

and ρ such that the $\Gamma_\rho^{i\bar{j}}$ has the appropriate number of indices. Obviously positivity of $H_N^{i\bar{j}}$ implies positivity of $H_1^{i\bar{j}}$, i.e. we only need to show the conservation property.

To identify an appropriate reduction to first order we first make the partial choice of reduction parameters

$$(D^X \sigma_\nu)^i = 0, \quad (\bar{D}^X \sigma_\nu)^i = 0, \quad (6.35)$$

for $\nu = 0, \dots, N-3$ and $\sigma = 1, \dots, N-\nu-2$, i.e. only the reduction parameters which correspond to the constraint additions with the highest number of derivative indices remain. As in (6.14) X denotes either a single index $\mu = 0, \dots, N-1$ or an index tuple $(\mu, \lambda, i_1, \dots, i_\lambda)$ with $\mu = 0, \dots, N-2$ and $\lambda = 1, \dots, N-\mu-1$.

With that choice most of the components of $\mathcal{A}_{1\bar{i}}^{p\bar{j}}$ vanish and the statement which needs to be shown is that there exist reduction parameters such that

$$H_N^{i\bar{j}} \tilde{\mathcal{A}}_{N\bar{j}}^{p\bar{k}} s_p \quad (6.36)$$

is Hermitian for every s , where

$$\tilde{\mathcal{A}}_{N\bar{j}}^{p\bar{k}} = \mathcal{A}_{N\bar{j}}^{p\bar{k}} + \bar{D}_{N\bar{j}}^{p\bar{k}} \quad (6.37)$$

and

$$\bar{D}_{N\bar{j}}^{p\bar{k}} = \begin{pmatrix} (\bar{D}^{\mu}_{N-\mu-1}{}^{N-\nu-1}{}_{\nu})_{\bar{j}}^{p\bar{k}} & 0 \\ (\bar{D}^{(N-1)}{}^{(N-\nu-1)}_{\nu})_{\bar{j}}^{p\bar{k}} & 0 \end{pmatrix}. \quad (6.38)$$

We define

$$(T_{\mu\nu}^{p\bar{i}\bar{k}})_{\mu=0, \dots, N-1}^{\nu=0, \dots, N-1} = T_N^{p\bar{i}\bar{k}} := H_N^{i\bar{j}} \mathcal{A}_{N\bar{j}}^{p\bar{k}}, \quad (6.39a)$$

$$(J_{\mu\nu}^{p\bar{i}\bar{k}})_{\mu=0, \dots, N-1}^{\nu=0, \dots, N-1} = J_N^{p\bar{i}\bar{k}} := H_N^{i\bar{j}} \bar{D}_{N\bar{j}}^{p\bar{k}}, \quad (6.39b)$$

where it is understood that decomposition of T_N and J_N into $T_{\mu\nu}$ and $J_{\mu\nu}$ is the one induced by the original FTNS system (6.1).

One finds that the hermiticity of (6.36) is equivalent to

$$T_N^{p\bar{i}\bar{k}} + J_N^{p\bar{i}\bar{k}} = \left(T_N^{p\bar{i}\bar{k}} + J_N^{p\bar{i}\bar{k}} \right)^\dagger \quad \forall p. \quad (6.40)$$

In components equation (6.40) is

$$J_{\mu\nu}^{p\ \underline{i}\ \underline{j}} + T_{\mu\nu}^{p\ \underline{i}\ \underline{j}} = J_{\nu\mu}^{\dagger p\ \underline{j}\ \underline{i}} + T_{\nu\mu}^{\dagger p\ \underline{j}\ \underline{i}} \quad \forall p. \quad (6.41)$$

From definition (6.39b) we see that the $J_{\mu\nu}$ need to satisfy certain symmetry conditions:

$$J_{\mu\nu}^{(p|\ \underline{i}\ |\underline{j})} = 0, \quad J_{\mu\nu}^{p\ \underline{i}\ \underline{j}} = J_{\mu\nu}^{p\ (\underline{i})\ (\underline{j})} \quad (6.42)$$

for $\mu = 0, \dots, N-1$, $\nu = 0, \dots, N-2$. Note that

$$J_{\mu\nu}^{(p|\ \underline{i}\ |\underline{j})} = 0 \quad \Rightarrow \quad J_{\mu(N-1)}^{p\ \underline{i}} = 0. \quad (6.43)$$

Since H_N is an FTNS candidate symmetrizer and due to the fact that certain symmetries hold for H_N and \mathcal{A}_N the $T_{\mu\nu}$ satisfy

$$T_{\mu\nu}^{(p|\ \underline{i}\ |\underline{j})} = T_{\nu\mu}^{\dagger(p\ \underline{j}\ \underline{i})}, \quad T_{\mu\nu}^{p\ \underline{i}\ \underline{j}} = T_{\mu\nu}^{p\ (\underline{i})\ (\underline{j})} \quad (6.44)$$

for $\mu, \nu = 0, \dots, N-1$.

Now, assuming a given $J_{\mu\nu}$ which satisfies (6.42) we can easily calculate the reduction variables \bar{D}_N by multiplication of J_N from the left with H_N^{-1} (which exists because H_N is positive definite by assumption).

Hence, the existence of a first order reduction with candidate symmetrizer H_1 is shown if we prove that there exist $J_{\mu\nu}$ which satisfy (6.41) and (6.42) given (6.44) holds.

One approach for the proof of this statement is the following. One defines

$$V_{\mu\nu}^{p\ \underline{i}\ \underline{j}} := T_{\mu\nu}^{p\ \underline{i}\ \underline{j}} - T_{\nu\mu}^{\dagger p\ \underline{j}\ \underline{i}}, \quad (6.45)$$

which satisfies

$$V_{\mu\nu}^{\dagger p\ \underline{i}\ \underline{j}} = -V_{\nu\mu}^{p\ \underline{j}\ \underline{i}}, \quad V_{\nu\mu}^{(p\ \underline{j}\ \underline{i})} = 0, \quad V_{\nu\mu}^{p\ \underline{j}\ \underline{i}} = V_{\nu\mu}^{p\ (\underline{j})\ (\underline{i})}. \quad (6.46)$$

Then one uses the ansatz

$$J_{\mu\nu}^{p\ \underline{i}\ \underline{j}} = \sum_{\pi \in S_{(2N-\mu-\nu-1)}} x_{\pi} V_{\mu\nu}^{\pi(p)\ \pi(\underline{i})\ \pi(\underline{j})} \quad (6.47)$$

in equations (6.41) and (6.42) to get a linear system for the coefficients x_{π} . If one can show that this linear system has a solution then the existence of a first order reduction with candidate symmetrizer H_1 follows with the arguments given above. This procedure is shown for $N = 3$ in appendix Appendix B and we performed the same calculations for $N \leq 4$ using computer algebra. In this *Mathematica* notebook we are using *xTensor* [21], it is available online [1, `automatic_construction_of_J.nb`].

For arbitrary N the number of coefficients increases like $N!$. Although many of them can be considered redundant because of the symmetries of $V_{\mu\nu}$ and $J_{\mu\nu}$, the construction of the linear system for the x_{π} is difficult for arbitrary N . Therefore we leave this question open.

Connection to energy conservation. Given an FTNS symmetric hyperbolic system it is straightforward to show that the quantity

$$E := \int d^3x u_i^\dagger H_N^{ij} u_j \quad (6.48)$$

is a conserved energy in the principal part, i.e. $E > 0$ and $\partial_t E \simeq 0$. To show this one uses the positivity of H_N^{ij} and the equations of motion (6.1) together with integration by parts:

$$\begin{aligned} \partial_t E &\simeq \frac{1}{2} \sum_{\mu, \nu=0}^{N-1} \int d^3x \dot{\epsilon}_{\mu\nu} \\ &= \frac{1}{2} \sum_{\mu, \nu=0}^{N-1} (-1)^{N-\mu-1} \int d^3x v^\dagger{}^\mu V_{\mu\nu}^{p\dot{i}\dot{j}} \partial_{p\dot{i}\dot{j}}^{(2N-\mu-\nu-1)} v^\nu = 0, \end{aligned} \quad (6.49)$$

where

$$\dot{\epsilon}_{\mu\nu} := \left(\partial_{\dot{i}}^{(N-\mu-1)} v^\dagger{}^\mu \right) T_{\mu\nu}^{p\dot{i}\dot{j}} \partial_{p\dot{j}}^{(N-\nu)} v^\nu + \left(\partial_{p\dot{i}}^{(N-\mu)} v^\dagger{}^\mu \right) T_{\nu\mu}^{\dagger p\dot{j}\dot{i}} \partial_{\dot{j}}^{(N-\nu-1)} v^\nu \quad (6.50)$$

and

$$\partial_{p\dot{i}\dot{j}}^{(2N-\mu-\nu-1)} v^\nu = \partial_p \partial_{i_1} \dots \partial_{i_{N-\mu-1}} \partial_{j_1} \dots \partial_{j_{N-\nu-1}} v^\nu. \quad (6.51)$$

Since v^μ is an arbitrary solution of the equations of motion this implies that there exist fluxes $\phi_{\mu\nu}^p$ such that

$$\dot{\epsilon}_{\mu\nu} = \partial_p \phi_{\mu\nu}^p \quad \forall \mu, \nu = 0, \dots, N-1. \quad (6.52)$$

The existence of a symmetric hyperbolic first order reduction with the symmetrizer H_1^{ij} means that there exist fluxes $\phi_{\mu\nu}^p$ of the form

$$\phi_{\mu\nu}^p = \left(\partial_{\dot{i}}^{(N-\mu-1)} v^\dagger{}^\mu \right) F_{\mu\nu}^{p\dot{i}\dot{j}} \partial_{\dot{j}}^{(N-\nu-1)} v^\nu, \quad (6.53)$$

with

$$F_{\mu\nu}^{p\dot{i}\dot{j}} = J_{\mu\nu}^{p\dot{i}\dot{j}} + T_{\mu\nu}^{p\dot{i}\dot{j}}, \quad (6.54)$$

i.e. that the v^μ appear in the fluxes only with $(N - \mu - 1)$ derivatives.

7. Conclusion

We described how the existing notion of strong hyperbolicity for first and second order in space evolution equations [14,11] can be extended to FTNS systems, i.e. evolution equations of arbitrary spatial order. The definitions of FTNS strong and symmetric hyperbolicity allow for the direct construction of well-posed initial (boundary) value problems for systems of higher order.

This extension is achieved by proposing a reasonable definition of strong hyperbolicity for FTNS systems and showing that this new definition can be reduced to the lower order equivalent. The proof is performed with the help of an iterative

differential reduction of the FTNS system from arbitrary to first order. One finds that an evolution system is FTNS strongly hyperbolic if and only if there exists a first order reduction which is strongly hyperbolic in the standard first order sense.

We also considered symmetric hyperbolicity of FTNS systems. In this case one finds that it is better to introduce a direct reduction to first order instead of using the iterative method applied to prove statements about strong hyperbolicity. We proposed a definition of FTNS symmetric hyperbolicity and were able to show for $N \leq 4$ that it is equivalent to the existence of a direct first order reduction which is symmetric hyperbolic in the standard first order sense. For higher orders we were not successful in showing equivalence, but only one direction, that the existence of a symmetric hyperbolic first order reduction implies FTNS symmetric hyperbolicity.

There are various questions which can be addressed in further analysis. One is that the proofs about strong hyperbolicity rely strongly on three spatial dimensions, because the Levi-Civita symbol ε_{ijk} is used. Whether a similar construction is possible for other spatial dimensionality is not known. For symmetric hyperbolicity the spatial dimensionality is not used in the calculations, i.e. the results apply to any dimension. However, as mentioned above, equivalence for $N > 4$ is not yet shown.

Finally, it is essential for the construction of approximate solutions to identify good numerical methods. Therefore it is also of interest to analyze the connection between high order hyperbolicity and e.g. stability of finite difference methods.

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Appendix A. Pseudo-differential reduction method

Reduction variables: To define strong hyperbolicity, in the literature a pseudo-differential reduction method is used, see for example [23]. With our calculations from section 5 it is straightforward to apply this method to FTNS systems as well. One takes a Fourier transformation in space of the FTNS system (4.4) with wave number $\omega_i = |\omega|s_i$. The Fourier transforms of the v^μ are denoted \hat{v}^μ and we introduce a reduction variable $\hat{d}^0 := i|\omega|\hat{v}^0$.

Principal part: Using the reduction variable the principal part of the Fourier transformed system (the terms with the highest order of $|\omega|$) can be written as

$$\partial_t \begin{pmatrix} (i|\omega|)^{N-2}\hat{d}^0 \\ (i|\omega|)^{N-\mu-1}\hat{v}^\mu \end{pmatrix} \simeq i|\omega|P_N^s \begin{pmatrix} (i|\omega|)^{N-2}\hat{d}^0 \\ (i|\omega|)^{N-\mu-1}\hat{v}^\mu \end{pmatrix}, \quad (\text{A.1})$$

where P_N^s is the principal symbol, (4.9), of the FTNS system (4.4) and the non principal terms not shown here are lower order in $|\omega|$. Applying this reduction $(N-1)$ times results in a first order pseudo-differential system with principal symbol P_N^s .

Hence, using definition 4b, an FTNS system is strongly hyperbolic if and only if there exists a strongly hyperbolic pseudo-differential reduction to order $(N - 1)$.

Appendix B. FT3S symmetric hyperbolicity through direct first order reduction

Problem statement: In this appendix we perform the main steps in showing that every FT3S candidate symmetrizer implies the existence of a direct first order reduction with associated candidate symmetrizer. That is, we perform the calculations which show that the system (3.54),(3.56) has a solution provided that (3.52) holds.

We start with repeating the system of interest in a compact form. With the definition (3.55), i.e.

$$V^{p\,ij\,kl} := T^{p\,ij\,kl} - T^{\dagger p\,kl\,ij} \quad (\text{B.1})$$

equation (3.52) gets

$$V^{(p\,ij\,kl)} = 0 \quad (\text{B.2})$$

and $V^{p\,ij\,kl}$ automatically satisfies

$$V^{p\,ij\,kl} = V^{p\,(ij)\,(kl)}. \quad (\text{B.3})$$

We are going to use these properties as assumptions in the proof that the system (3.54),(3.56), i.e.

$$J^{p\,ij\,kl} = J^{p\,(ij)\,(kl)}, \quad J^{(p\,ij\,kl)} = 0, \quad J^{p\,ij\,kl} - J^{\dagger p\,kl\,ij} = -V^{p\,ij\,kl}, \quad (\text{B.4})$$

has a solution.

We do this by explicit construction using the decomposition

$$J^{p\,ij\,kl} = \begin{pmatrix} J_{uu}^{p\,ij\,kl} & J_{uv}^{p\,ij\,k} & J_{uw}^{p\,ij} \\ J_{vu}^{p\,i\,kl} & J_{vv}^{p\,i\,k} & J_{vw}^{p\,i} \\ J_{wu}^{p\,kl} & J_{wv}^{p\,k} & J_{ww}^{p\,} \end{pmatrix}. \quad (\text{B.5})$$

The construction procedure is performed for each component in this decomposition separately.

uw-, vw- and ww-component: We start with the most simple cases, J_{uw} , J_{vw} and J_{ww} . As we can read off from definition (B.1), the following equations hold

$$V_{uw}^{\dagger p\,ij} = -V_{wu}^{p\,ij}, \quad V_{vw}^{\dagger p\,i} = -V_{wu}^{p\,i}, \quad V_{ww}^{\dagger p} = -V_{wu}^{p\,}. \quad (\text{B.6})$$

Using this and choosing

$$J_{ww}^p = 0, \quad J_{vw}^{p\,i} = 0, \quad J_{wv}^{p\,k} = -V_{wv}^{p\,k}, \quad J_{uw}^{p\,ij} = 0, \quad J_{wu}^{p\,kl} = -V_{wu}^{p\,kl} \quad (\text{B.7})$$

one finds that

$$\begin{aligned}
J_{ww}^p - J_{ww}^{\dagger p} &= 0 = -V_{ww}^{(p)} = -V_{ww}^p, & J_{vv}^{pi} - J_{vv}^{\dagger pi} &= V_{vv}^{\dagger pi} = -V_{vv}^{pi}, \\
J_{wv}^{pi} - J_{wv}^{\dagger pi} &= -V_{wv}^{pi}, & J_{uw}^{ij} - J_{uw}^{\dagger ij} &= V_{uw}^{\dagger ij} = -V_{uw}^{ij}, \\
J_{wu}^{ij} - J_{wu}^{\dagger ij} &= -V_{wu}^{ij}, & J_{wv}^{(pk)} &= -V_{wv}^{(pk)} = 0, \\
J_{wu}^{(pk)} &= -V_{wu}^{(pk)} = 0.
\end{aligned} \tag{B.8}$$

Hence, (B.7) is a solution of the uw -, vw - and ww -components of (B.4).

uu -component: Solving for the uu -component of J involves the most lengthy calculations. Using again the FT3S conservation property

$$V_{uu}^{(p\ ij\ kl)} = 0 \tag{B.9}$$

one finds that one solution of the UU -component of (B.4) is

$$\begin{aligned}
J_{uu}^{p\ ij\ kl} &= -1/2V_1^{p\ ij\ kl} - 9/20V_4^{p\ ij\ kl} - 3/20V_5^{p\ ij\ kl} - 1/20V_6^{p\ ij\ kl} - 7/20V_7^{p\ ij\ kl} \\
&\quad + 1/20V_8^{p\ ij\ kl} + 7/20V_9^{p\ ij\ kl} + 9/20V_{10}^{p\ ij\ kl} + 3/20V_{11}^{p\ ij\ kl},
\end{aligned} \tag{B.10}$$

with

$$\begin{aligned}
V_1^{p\ ij\ kl} &:= V_{uu}^{p\ ij\ kl}, \\
V_2^{p\ ij\ kl} &:= V_{uu}^{p\ kl\ ij}, \\
4V_3^{p\ ij\ kl} &:= V_{uu}^{p\ ik\ jl} + V_{uu}^{p\ jk\ il} + V_{uu}^{p\ il\ jk} + V_{uu}^{p\ jl\ ik}, \\
2V_4^{p\ ij\ kl} &:= V_{uu}^{i\ pj\ kl} + V_{uu}^{j\ pi\ kl}, \\
4V_5^{p\ ij\ kl} &:= V_{uu}^{i\ pk\ jl} + V_{uu}^{j\ pk\ il} + V_{uu}^{i\ pl\ jk} + V_{uu}^{j\ pl\ ik}, \\
2V_6^{p\ ij\ kl} &:= V_{uu}^{k\ pl\ ij} + V_{uu}^{l\ pk\ ij}, \\
4V_7^{p\ ij\ kl} &:= V_{uu}^{k\ pi\ jl} + V_{uu}^{l\ pj\ il} + V_{uu}^{l\ pi\ jk} + V_{uu}^{l\ pj\ ik}, \\
2V_8^{p\ ij\ kl} &:= V_{uu}^{i\ kl\ pj} + V_{uu}^{j\ kl\ pi}, \\
4V_9^{p\ ij\ kl} &:= V_{uu}^{i\ jk\ pl} + V_{uu}^{j\ ik\ pl} + V_{uu}^{i\ jl\ pk} + V_{uu}^{j\ il\ pk}, \\
2V_{10}^{p\ ij\ kl} &:= V_{uu}^{k\ ij\ pl} + V_{uu}^{l\ ij\ pk}, \\
4V_{11}^{p\ ij\ kl} &:= V_{uu}^{k\ il\ pj} + V_{uu}^{l\ jl\ pi} + V_{uu}^{l\ ik\ pj} + V_{uu}^{l\ jk\ pi}.
\end{aligned} \tag{B.11}$$

The solution (B.10) is not unique. If J_{uu} solves the uu -component in (B.4) then $J_{uu}^{p\ ij\ kl} + W_{uu}^{p\ ij\ kl}$ is also a solution for every W_{uu} that satisfies

$$W_{uu}^{\dagger p\ kl\ ij} = W_{uu}^{p\ ij\ kl}, \quad W_{uu}^{p\ |ij|\ kl} = 0 = W_{uu}^{(p\ kl)\ ij}, \quad W_{uu}^{p\ kl\ ij} = W_{uu}^{p\ (kl)\ (ij)}.$$

To construct (B.10) we make the ansatz that J_{uu} is a linear combination of V_{uu} with permuted indices. Since $J_{uu}^{p\ ij\ kl}$ needs to be symmetric in (i, j) and (k, l) , and since also $V_{uu}^{p\ ij\ kl}$ is automatically symmetric in (i, j) and (k, l) the number of independent coefficients reduces to 11:

$$J_{uu}^{p\ ij\ kl} = \sum_{N=1}^{11} x_N V_N^{p\ ij\ kl}, \tag{B.12}$$

where the V_N can be read off from (B.11). Note that

$$\begin{aligned} V_1^{\dagger p kl ij} &= -V_1^{p ij kl}, & V_2^{\dagger p kl ij} &= -V_2^{p ij kl}, & V_3^{\dagger p kl ij} &= -V_3^{p ij kl}, \\ V_4^{\dagger p kl ij} &= -V_{10}^{p ij kl}, & V_5^{\dagger p kl ij} &= -V_{11}^{p ij kl}, & V_6^{\dagger p kl ij} &= -V_8^{p ij kl}, \\ V_7^{\dagger p kl ij} &= -V_9^{p ij kl}. \end{aligned} \quad (\text{B.13})$$

Using the ansatz (B.12) together with (B.13) in the uu -component of the last of equations (B.4) one finds a linear system for the coefficients x_i . The solution of this system restrict the possible choices of the coefficients to

$$\begin{aligned} x_1 &= -1/2, & x_2 &= 0, & x_3 &= 0, & x_8 &= -x_6, \\ x_9 &= -x_7, & x_{10} &= -x_4, & x_{11} &= -x_5. \end{aligned} \quad (\text{B.14})$$

Here we did not rely on the conservation equation (B.2). Analogously, using (B.12) and (B.14) in the second equation of (B.4) one finds that making the choice

$$x_4 = -\frac{9}{20}, \quad x_5 = -\frac{3}{20}, \quad x_6 = -\frac{1}{20}, \quad x_7 = -\frac{7}{20}. \quad (\text{B.15})$$

leads to

$$J_{uu}^{(p|ij|kl)} = -1/2 V_{uu}^{(p ij kl)}. \quad (\text{B.16})$$

One can verify this by using

$$\begin{aligned} V^{(p ij kl)} &= \frac{1}{30} (V_1^{p ij kl} + V_2^{p ij kl} + 4V_3^{p ij kl} + 2V_4^{p ij kl} + 4V_5^{p ij kl} + 2V_6^{p ij kl} \\ &\quad + 4V_7^{p ij kl} + 2V_8^{p ij kl} + 4V_9^{p ij kl} + 2V_{10}^{p ij kl} + 4V_{11}^{p ij kl}). \end{aligned} \quad (\text{B.17})$$

Hence, with the choice (B.14) and (B.15) the ansatz (B.12) results in the solution (B.10) of the uu -component of (B.4) provided that the uu -component of (B.2) holds.

vv -component: For the vv -component the calculations were already done in [11]. The approach is the same as for the uu -component. One finds that

$$J_{vv}^{p ik} = -1/2 V_{vv}^{p ik} + 1/6 V_{vv}^{i kp} - 1/3 V_{vv}^{i p k} + 1/3 V_{vv}^{k ip} \quad (\text{B.18})$$

solves the vv -component of (B.4) provided that the vv -component of (B.2) is satisfied.

uv -component conservation: Finally one can show in a similar way that

$$J_{uv}^{p ij k} = -3/4 V_{uv}^{p ij k} + 1/2 V_{uv}^{p k(ij)} + 3/4 V_{uv}^{k ij p} - 1/2 V_{uv}^{k p(ij)}, \quad (\text{B.19a})$$

$$J_{vu}^{p k ij} = -1/4 V_{vu}^{p k ij} - 1/2 V_{vu}^{p(ij)k} - 3/4 V_{vu}^{p k ij} + 1/2 V_{vu}^{k(ij)p} \quad (\text{B.19b})$$

is the desired solution for the uv -component of (B.4).

Here it is interesting that one does not need to start from the most general ansatz that J_{uv} and J_{vu} are linear combinations of both, V_{uv} and V_{vu} , but that a

more restricting ansatz is sufficient:

$$J_{uv}^{p\,ij\,k} = x_1 V_{uv}^{p\,ij\,k} + x_2 V_{uv}^{p\,k(ij)} + x_3 V_{uv}^{k\,ij\,p} + x_4 V_{uv}^{k\,p(ij)} + x_5 V_{uv}^{(ij)k\,p} + x_6 V_{uv}^{(ij)p\,k} + x_7 V_{uv}^{(i|p\,k|j)}, \quad (\text{B.20a})$$

$$J_{vu}^{p\,ij\,k} = x_8 V_{vu}^{p\,k\,ij} + x_9 V_{vu}^{p(ij)\,k} + x_{10} V_{vu}^{k\,p\,ij} + x_{11} V_{vu}^{k(ij)\,p} + x_{12} V_{vu}^{(i|p\,k|j)} + x_{13} V_{vu}^{(i|k\,p|j)} + x_{14} V_{vu}^{(ij)\,p\,k}. \quad (\text{B.20b})$$

The basic reason for that is

$$V_{uv}^{\dagger p\,ij\,k} = -V_{vu}^{\dagger p\,k\,ij}. \quad (\text{B.21})$$

Using this ansatz as before in (B.4) one finds that (B.19) satisfies

$$J_{uv}^{p\,ij\,k} = J_{uv}^{p(ij)\,k}, \quad J_{uv}^{p(ij)\,k} = 0, \quad J_{vu}^{(p|k|ij)} = V_{vu}^{(p\,k\,ij)}.$$

Altogether this shows that each FT3S candidate symmetrizer implies the existence of a direct first order reduction with associated candidate symmetrizer. \square

The above calculations are presented in completeness in the `Mathematica` notebook `flux.construction.nb` which can be found online [1].

Appendix C. Problem in construction of FT2S symmetric hyperbolic reductions

In section 3.4 we mentioned that we cannot use FT3S symmetrizers to construct symmetrizers for FT2S reductions. Here we explain why this approach does not work.

C.1. Simple construction of FT2S symmetrizers

Given an FT3S symmetrizer, $H_3^{ia\,jb}$, the natural first step in the construction of an FT2S symmetrizer, $H_2^{ij\,ab}$, is the ansatz

$$H_2^{ij\,ab} = \begin{pmatrix} \gamma^{ij} & 0 \\ 0 & H_3^{ia\,jb} \end{pmatrix}. \quad (\text{C.1})$$

FT2S conservation equations: The terms which contain reduction parameters in the product of H_2 with the FT2S principal matrix, $H_2^{ij\,ab} \mathcal{A}_{2j}^{p\,k\,c}$, are given by

$$\tilde{J}^{p\,ik\,ac} = \begin{pmatrix} \gamma^{ip} & 0 & 0 & 0 \\ 0 & H_{uu}^{ia\,pb} & H_{uv}^{ia\,p} & H_{uw}^{ia} \\ 0 & H_{vu}^{i\,pb} & H_{vv}^{i\,p} & H_{vw}^i \\ 0 & H_{wu}^{pb} & H_{wv}^p & H_{ww} \end{pmatrix} \begin{pmatrix} (D^u)^k & (\bar{D}^u)^{kc} & 0 & 0 \\ (D)_b^k & (\bar{D})_b^{kc} & 0 & 0 \\ (D^v)^k & (\bar{D}^v)^{kc} & 0 & 0 \\ (D^w)^{pk} & (\bar{D}^w)^{pkc} & 0 & 0 \end{pmatrix}, \quad (\text{C.2})$$

where we already contracted a couple of δ 's. If there exists an FT2S reduction which has $H_2^{ij\,ab}$ as a symmetrizer then, according to the definitions in section 2 and the antisymmetry of the reduction variables, the following equations are satisfied

$$\tilde{J}^{(p\,ik)\,ac} - \tilde{J}^{\dagger(p\,ki)\,ca} = \tilde{V}^{(p\,ik)\,ac}, \quad \tilde{J}^{p\,ik\,ac} = -\tilde{J}^{p\,ic\,ak}, \quad (\text{C.3})$$

where

$$\tilde{V}^{p\,ik\,ac} = H_2^{ij\,ab} \tilde{\mathcal{A}}_{2j}^{p\,k\,c} - \tilde{\mathcal{A}}_2^{\dagger p\,i\,a} H_2^{jk\,bc} \quad (\text{C.4})$$

and

$$\tilde{\mathcal{A}}_{2j}^{p\,k\,c} = \mathcal{A}_{2j}^{p\,k\,c} \big|_{D=0, \bar{D}=0}. \quad (\text{C.5})$$

Assume that (C.3) has a solution

$$\tilde{J}^{p\,ik\,ac} = \bar{V}^{p\,i[k|a|c]}, \quad (\text{C.6})$$

where $\bar{V}^{p\,ik\,ac}$ is a matrix that does not contain D or \bar{D} . To find the corresponding solution for the reduction parameters one multiplies (C.6) from the left by the inverse of $H_2^{ij\,ab}$ and gets

$$H_2^{-1}{}_{ji\,ba} \tilde{J}^{p\,ik\,ac} = \begin{pmatrix} \delta_j^p(D^u)^k & \delta_j^p(\bar{D}^u)^{kc} & 0 & 0 \\ \delta_j^p(D)^b{}^k & \delta_j^p(\bar{D})_b{}^{kc} & 0 & 0 \\ \delta_j^p(D^v)^k & \delta_j^p(\bar{D}^v)^{kc} & 0 & 0 \\ (D^w)^{pk} & (\bar{D}^w)^{pkc} & 0 & 0 \end{pmatrix} \stackrel{!}{=} H_2^{-1}{}_{ji\,ba} \bar{V}^{p\,i[k|a|c]}. \quad (\text{C.7})$$

This implies that $H_2^{-1}{}_{ji\,ba} \bar{V}^{p\,i[k|a|c]}$ has the same structure as $H_2^{-1}{}_{ji\,ba} \tilde{J}^{p\,ik\,ac}$. In particular the $(2,2)$ -component of this product must be of the form $\delta_j^p \bar{V}_{22b}{}^{kc}$ and the last two columns need to vanish. The counterexample given in appendix C.2 shows that this is not always the case, i.e. there are FT3S systems for which there exists no FT2S reduction with the symmetrizer $H_2^{ij\,ab}$. This does not mean that there is no symmetric hyperbolic FT2S reduction, but if there is one then its symmetrizer has no direct relation to the given FT3S symmetrizer.

Discussion: The problems in the construction of a second order conserved energy from a given third order symmetrizer can be attributed to the fact that the product of H_2^{-1} and \tilde{J} contain δ_j^p . If e.g. the $(2,2)$ -component of this matrix was not $\delta_j^p(\bar{D})_b{}^{kc}$, but $(\bar{D})_{jb}{}^{pkc}$ then this would not restrict the form of the $(2,2)$ -component of $H_2^{-1}{}_{ji\,ba} \bar{V}^{p\,i[k|a|c]}$. Thus, in order to avoid these problems we need reduction parameters with two lower indices. This can be achieved by using a direct reduction to first order and construct a first order symmetrizer from $H_3^{ia\,jb}$, as described in section 3.6.

C.2. Not every FT3S candidate has an associated FT2S candidate

In what follows we present an example where the conserved energy of a third order system is not the energy of any second order reduction in the sense of section 3.2. The calculations are done in the notebook `counter_example_3rd_order_sym_hyp.nb` which is available online [1].

Counterexample: Our counterexample is the following FT3S system

$$\begin{aligned}
\partial_t u &= a_{uu}^x \partial_x u + a_{uu}^y \partial_y u + a_{uv} v, \\
\partial_t v &= a_{vu}^{xx} \partial_x \partial_x u + a_{vu}^{xy} \partial_x \partial_y u + a_{vu}^{yy} \partial_y \partial_y u + a_{vv}^x \partial_x v + a_{vv}^y \partial_y v + a_{vw} w, \\
\partial_t w &= a_{wu}^{xxx} \partial_x \partial_x \partial_x u + a_{wu}^{xxy} \partial_x \partial_x \partial_y u + a_{wu}^{xyy} \partial_x \partial_y \partial_y u + a_{wu}^{yyy} \partial_y \partial_y \partial_y u + a_{wv}^{xx} \partial_x \partial_x v \\
&\quad + a_{wv}^{xy} \partial_x \partial_y v + a_{wv}^{yy} \partial_y \partial_y v + a_{ww}^x \partial_x w + a_{ww}^y \partial_y w,
\end{aligned} \tag{C.8}$$

with

$$\begin{aligned}
a_{uu}^x &= \frac{17273}{7384}, & a_{uu}^y &= 1, & a_{uv} &= -\frac{12951}{14768}, \\
a_{vu}^{xx} &= -\frac{2855}{1846}, & a_{vu}^{xy} &= -\frac{8575}{44304}, & a_{vu}^{yy} &= -\frac{1095}{3692}, \\
a_{vv}^x &= 1, & a_{vv}^y &= \frac{1}{2}, & a_{vw} &= 1, \\
a_{wu}^{xxx} &= \frac{1}{2}, & a_{wu}^{xxy} &= \frac{1}{9}, & a_{wu}^{xyy} &= 0, \\
a_{wu}^{yyy} &= \frac{1}{3}, & a_{wv}^{xx} &= 1, & a_{wv}^{xy} &= \frac{1}{10}, \\
a_{wv}^{yy} &= 1, & a_{ww}^x &= 1, & a_{ww}^y &= 1.
\end{aligned} \tag{C.9}$$

It has the conserved energy

$$E_3 = \int d^2x \left(\partial_i \partial_j u, \partial_i v, w \right) H^{ij\,kl} \begin{pmatrix} \partial_k \partial_l u \\ \partial_k v \\ w \end{pmatrix}, \tag{C.10}$$

with

$$H^{ij\,kl} = \begin{pmatrix} H_{uu}^{ij\,kl} & H_{uv}^{ij\,k} & H_{uw}^{ij} \\ H_{uv}^{\dagger kl\,i} & H_{vv}^{\dagger k} & H_{vw}^{\dagger i} \\ H_{uw}^{\dagger kl} & H_{vw}^{\dagger k} & H_{ww} \end{pmatrix}. \tag{C.11}$$

In the upper left we have

$$H_{uu}^{ij\,kl} = \begin{pmatrix} h_{uu}^{xxxx} & h_{uu}^{xxxy} & h_{uu}^{xxyx} & h_{uu}^{xxyy} \\ h_{uu}^{xxxy} & h_{uu}^{xxyx} & h_{uu}^{xxyy} & h_{uu}^{xyyy} \\ h_{uu}^{xxyx} & h_{uu}^{xxyy} & h_{uu}^{xyyy} & h_{uu}^{yyyy} \\ h_{uu}^{xyyy} & h_{uu}^{yyyy} & h_{uu}^{yyyy} & h_{uu}^{yyyy} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4/27 \end{pmatrix},$$

next the off-diagonal components

$$H_{uv}^{ij\,k} = \begin{pmatrix} h_{uv}^{xx} & h_{uv}^{xy} \\ h_{uv}^{xy} & h_{uv}^{yy} \\ h_{uv}^{yy} & h_{uv}^{yy} \\ h_{uv}^{yy} & h_{uv}^{yy} \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 1/3 & 0 \\ 0 & 0 \\ 0 & 1/3 \end{pmatrix},$$

the vanishing terms

$$H_{uw}^{ij} = \begin{pmatrix} h_{uw}^{xx} \\ h_{uw}^{xy} \\ h_{uw}^{yx} \\ h_{uw}^{yy} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and the simple remaining

$$\begin{aligned} H_{vv}^{ik} &= \begin{pmatrix} h_{vv}^{xx} & h_{vv}^{xy} \\ h_{vv}^{xy} & h_{vv}^{yy} \end{pmatrix} = \begin{pmatrix} 1 & 1/10 \\ 1/10 & 1 \end{pmatrix}, & H_{vw}^i &= (h_{vw}^x \ h_{vw}^y) = (0 \ 0), \\ H_{ww} &= 1. \end{aligned} \tag{C.12}$$

Nonexistence of an FT2S symmetrizer: Given the system (C.8) and the symmetrizer (C.11) the construction of a second order reduction with symmetrizer

$$\begin{pmatrix} \gamma^{ik} & 0 \\ 0 & H^{ij\ kl} \end{pmatrix} \tag{C.13}$$

involves only solving linear equations. One finds that the corresponding linear system is overdetermined and has no solution for the above example. Hence, there is no reduction to second order such that the following expression is conserved

$$\int d^2x (\partial_i u, \partial_i d_j, \partial_i v, w) \begin{pmatrix} \gamma^{ik} & 0 \\ 0 & H^{ij\ kl} \end{pmatrix} \begin{pmatrix} \partial_k u \\ \partial_k d_l \\ \partial_k v \\ w \end{pmatrix}. \tag{C.14}$$

This differs from the case of reductions of second order systems to first order systems. There every second order energy is also the energy of a certain first order reduction.

Appendix D. Auxiliary constraint evolution for direct first order reductions

In section 6.1 we introduced a direct reduction of FTNS systems to first order. The calculations to derive the auxiliary constraint evolution system for this reduction are shown here.

Evolution of the c_ν^μ : We start with the constraints that allow one to change the derivative order of the equation. For $\mu = 0, \dots, N-3$ and $\nu = 1, \dots, N-\mu-2$ one

finds

$$\begin{aligned}
\partial_t(c_\nu^\mu)_{\underline{i}} &= \partial_t \partial_{(i_1}(d_{\nu-1}^\mu)_{i_2 \dots i_\nu}) - \partial_t(d_\nu^\mu)_{i_1 \dots i_\nu} \\
&= \sum_{\sigma=0}^{\mu+1} (A^\mu_\sigma)^{\underline{j}} [\partial_{(i_1}(d_{\mu+\nu-\sigma}^\sigma)_{i_2 \dots i_\nu})_{\underline{j}} - (d_{\mu+\nu-\sigma+1}^\sigma)_{i_1 \dots i_\nu \underline{j}}] \\
&\quad + \sum_{\sigma=0}^{\mu} \sum_{\rho=1}^{\mu-\sigma+1} (B^\mu_{\rho\sigma})^{\underline{j}} [\partial_{(i_1}(d_{\mu+\nu-\sigma-\rho}^\sigma)_{i_2 \dots i_\nu})_{\underline{j}} - (d_{\mu+\nu-\sigma-\rho+1}^\sigma)_{i_1 \dots i_\nu \underline{j}}] \\
&\quad + \partial_{(i_1}(C_{\nu-1}^\mu)_{i_2 \dots i_\nu}) - (C_\nu^\mu)_{i_1 \dots i_\nu} \\
&= \sum_{\sigma=0}^{\mu+1} (A^\mu_\sigma)^{\underline{j}} [(c_{\mu+\nu-\sigma+1}^\sigma)_{i_1 \dots i_\nu \underline{j}} + (\bar{c}_{\mu+\nu-\sigma}^\sigma)_{(i_1 \dots i_\nu) \underline{j}}] \\
&\quad + \sum_{\sigma=0}^{\mu} \sum_{\rho=1}^{\mu-\sigma+1} (B^\mu_{\rho\sigma})^{\underline{j}} [(c_{\mu+\nu-\sigma-\rho+1}^\sigma)_{i_1 \dots i_\nu \underline{j}} \\
&\quad + (\bar{c}_{\mu+\nu-\sigma-\rho}^\sigma)_{(i_1 \dots i_\nu) \underline{j}}] + \partial_{(i_1}(C_{\nu-1}^\mu)_{i_2 \dots i_\nu}) - (C_\nu^\mu)_{i_1 \dots i_\nu},
\end{aligned} \tag{D.1a}$$

where for $\nu = 1$ we identified $C_0^\mu := C^\mu$ and $d_0^\mu = v^\mu$. The source terms containing s^μ obviously cancel. Next we do the same calculation for $\mu = 0, \dots, N-2$ and $\nu = N - \mu - 1$. The main difference is that the terms which contain A^μ_ν include a derivative which is expressed differently in terms of the constraints. One finds

$$\begin{aligned}
\partial_t(c_{N-\mu-1}^\mu)_{\underline{i}} &= \partial_t \partial_{(i_1}(d_{N-\mu-2}^\mu)_{i_2 \dots i_{N-\mu-1}}) - \partial_t(d_{N-\mu-1}^\mu)_{i_1 \dots i_{N-\mu-1}} \\
&= \sum_{\sigma=0}^{\min(\mu+1, N-2)} (A^\mu_\sigma)^{\underline{j}} [\partial_{(i_1}(d_{N-\mu-1}^\sigma)_{i_2 \dots i_{N-\mu-1}})_{\underline{j}} - \partial_{(i_1}(d_{N-\mu-1}^\sigma)_{i_2 \dots i_{N-\mu-1} \underline{j}})] \\
&\quad + \sum_{\sigma=0}^{\mu} \sum_{\rho=1}^{\mu-\sigma+1} (B^\mu_{\rho\sigma})^{\underline{j}} [\partial_{(i_1}(d_{N-\sigma-\rho-1}^\sigma)_{i_2 \dots i_{N-\mu-1}})_{\underline{j}} - (d_{N-\sigma-\rho}^\sigma)_{i_1 \dots i_{N-\mu-1} \underline{j}}] \\
&\quad + \partial_{(i_1}(C_{N-\mu-2}^\mu)_{i_2 \dots i_{N-\mu-1}}) - (C_{N-\mu-1}^\mu)_{i_1 \dots i_{N-\mu-1}} \\
&= \sum_{\sigma=0}^{\min(\mu+1, N-2)} (A^\mu_\sigma)^{\underline{j}} (\bar{c}_{N-\sigma-1}^\sigma)_{(i_1 \dots i_{N-\mu-1}) \underline{j}} \\
&\quad + \sum_{\sigma=0}^{\mu} \sum_{\rho=1}^{\mu-\sigma+1} (B^\mu_{\rho\sigma})^{\underline{j}} [(c_{N-\sigma-\rho}^\sigma)_{i_1 \dots i_{N-\mu-1} \underline{j}} + (\bar{c}_{N-\sigma-\rho-1}^\sigma)_{(i_1 \dots i_{N-\mu-1}) \underline{j}}] \\
&\quad + \partial_{(i_1}(C_{N-\mu-2}^\mu)_{i_2 \dots i_{N-\mu-1}}) - (C_{N-\mu-1}^\mu)_{i_1 \dots i_{N-\mu-1}},
\end{aligned} \tag{D.1b}$$

where for $\mu = N-2$ the terms containing v^{N-1} cancel and we identify $C_0^{N-2} := C^{N-2}$ and $d_0^{N-2} = v^{N-2}$.

Evolution of the \bar{c}_ν^μ : Now we show the analogous statement for the \bar{c}_ν^μ . To make the resulting expressions shorter we use the following notation for the non symmetric

part of a tensor X

$$X_{i_1 \dots \{i_\alpha \dots i_\beta\} \dots i_\gamma} := X_{i_1 \dots i_\alpha \dots i_\beta \dots i_\gamma} - X_{i_1 \dots (i_\alpha \dots i_\beta) \dots i_\gamma} =: \text{usym}_{i_\alpha, \dots, i_\beta}(X_{i_1 \dots i_\alpha \dots i_\beta \dots i_\gamma}). \quad (\text{D.2})$$

Furthermore we use the following notation for derivatives of the constraints

$$\begin{aligned} (\partial^\lambda c_\nu^\mu)_{i_1 \dots i_{\nu+\lambda}} &:= \partial_{i_1} \dots \partial_{i_\lambda} (c_\nu^\mu)_{i_{\lambda+1} \dots i_{\nu+\lambda}}, \\ (\partial^\lambda \bar{c}_\nu^\mu)_{i_1 \dots i_{\nu+\lambda+1}} &:= \partial_{i_1} \dots \partial_{i_\lambda} (\bar{c}_\nu^\mu)_{i_{\lambda+1} \dots i_{\nu+\lambda+1}}. \end{aligned} \quad (\text{D.3})$$

Again we start with the cases where $\mu = 0, \dots, N-3$ and $\nu = 0, \dots, N-\mu-2$ and find

$$\begin{aligned} \partial_t (\bar{c}_\nu^\mu)_{\underline{j}} &= \partial_t \partial_{\{i_1} (d_\nu^\mu)_{i_2 \dots i_{\nu+1}\}} \quad (\text{D.4a}) \\ &= \partial_{\{i_1} (C_\nu^\mu)_{i_2 \dots i_{\nu+1}\}} + \sum_{\sigma=0}^{\mu+1} (A^\mu_\sigma)^{\underline{j}} \partial_{\{i_1} (d_{\mu+\nu-\sigma+1}^\sigma)_{i_2 \dots i_{\nu+1}\}}_{\underline{j}} \\ &\quad + \sum_{\sigma=0}^{\mu} \sum_{\rho=1}^{\mu-\sigma+1} (B^\mu_{\rho\sigma})^{\underline{j}} \partial_{\{i_1} (d_{\mu+\nu-\sigma-\rho+1}^\sigma)_{i_2 \dots i_{\nu+1}\}}_{\underline{j}} \\ &= \partial_{\{i_1} (C_\nu^\mu)_{i_2 \dots i_{\nu+1}\}} \\ &\quad - \sum_{\sigma=0}^{\mu+1} \sum_{\lambda=0}^{\mu+\nu-\sigma} (A^\mu_\sigma)^{\underline{j}} (\partial^{\lambda+1} c_{\mu+\nu-\sigma-\lambda+1}^\mu)_{\{i_1 \dots i_{\nu+1}\}}_{\underline{j}} \\ &\quad - \sum_{\sigma=0}^{\mu+1} \sum_{\lambda=0}^{\mu+\nu-\sigma-1} (A^\mu_\sigma)^{\underline{j}} (\partial^{\lambda+1} \bar{c}_{\mu+\nu-\sigma-\lambda}^\mu)_{\{i_1 \dots i_{\nu+1}\}}_{\underline{j}} \\ &\quad - \sum_{\sigma=0}^{\mu} \sum_{\rho=1}^{\mu-\sigma+1} \sum_{\lambda=0}^{\mu+\nu-\sigma-\rho} (B^\mu_{\rho\sigma})^{\underline{j}} (\partial^{\lambda+1} c_{\mu+\nu-\sigma-\rho-\lambda+1}^\mu)_{\{i_1 \dots i_{\nu+1}\}}_{\underline{j}} \\ &\quad - \sum_{\sigma=0}^{\mu} \sum_{\rho=1}^{\mu-\sigma+1} \sum_{\lambda=0}^{\mu+\nu-\sigma-\rho-1} (B^\mu_{\rho\sigma})^{\underline{j}} (\partial^{\lambda+1} \bar{c}_{\mu+\nu-\sigma-\rho-\lambda}^\mu)_{\{i_1 \dots i_{\nu+1}\}}_{\underline{j}}. \end{aligned}$$

There we used (6.8) to express the reduction variables in terms of constraints and spatial derivatives of the v^μ . Since we consider only the non symmetric part of the expression the terms which contain v^μ cancel and one finds the equation above.

Finally, the case $\mu = 0, \dots, N-2$ and $\nu = N-\mu-1$ is shown analogously. The only difference is that the right hand side contains terms with derivatives of higher

order acting on the reduction variables:

$$\begin{aligned}
\partial_t(\bar{c}_{N-\mu}^\mu)_{\underline{i}} &= \partial_t \partial_{\{i_1}(d^\mu)_{i_2 \dots i_{N-\mu}\}} \tag{D.4b} \\
&= \partial_{\{i_1}(C_{N-\mu-1}^\mu)_{i_2 \dots i_{N-\mu}\}} + \text{usym}_{i_1 \dots i_{N-\mu}} \left(\sum_{\sigma=0}^{\min(\mu+1, N-2)} (A^\mu)_\sigma \bar{c}_{i_1}^j \partial_{(i_2}(d^\sigma)_{i_3 \dots i_{N-\mu}j}) \right) \\
&\quad + \sum_{\sigma=0}^{\mu} \sum_{\rho=1}^{\mu-\sigma+1} (B^\mu)_{\rho\sigma} \bar{c}_{i_1}^j \partial_{\{i_1}(d_{N-\sigma-\rho}^\sigma)_{i_2 \dots i_{N-\mu}\}} \bar{c}_{j\}} \\
&= \partial_{\{i_1}(C_{N-\mu-1}^\mu)_{i_2 \dots i_{N-\mu}\}} \\
&\quad - \text{usym}_{i_1 \dots i_{N-\mu}} \left(\sum_{\sigma=0}^{\min(\mu+1, N-2)} \sum_{\lambda=0}^{N-\sigma-2} (A^\mu)_\sigma \bar{c}_{i_1}^j (\partial^{\lambda+2} c_{N-\sigma-\lambda-1}^\sigma)_{i_2 \dots i_{N-\mu}j} \right) \\
&\quad - \text{usym}_{i_1 \dots i_{N-\mu}} \left(\sum_{\sigma=0}^{\mu+1} \sum_{\lambda=0}^{N-\sigma-3} (A^\mu)_\sigma \bar{c}_{i_1}^j (\partial^{\lambda+2} \bar{c}_{N-\sigma-\lambda-2}^\sigma)_{i_2 \dots i_{N-\mu}j} \right) \\
&\quad - \sum_{\sigma=0}^{\mu} \sum_{\rho=1}^{\mu-\sigma+1} (B^\mu)_{\rho\sigma} \bar{c}_{i_1}^j \left(\sum_{\lambda=0}^{N-\sigma-\rho-1} (\partial^{\lambda+1} c_{N-\sigma-\rho-\lambda}^\sigma)_{\{i_1 \dots i_{N-\mu}\}} \bar{c}_{j\}} \right. \\
&\quad \quad \left. + \sum_{\lambda=0}^{N-\sigma-\rho-2} (\partial^{\lambda+1} \bar{c}_{N-\sigma-\rho-\lambda-1}^\sigma)_{\{i_1 \dots i_{N-\mu}\}} \bar{c}_{j\}} \right).
\end{aligned}$$

Hence, we derived an explicit expression for the constraint evolution system which shows that this system is closed.

References

- [1] <http://www.tpi.uni-jena.de/~hild/FTNS.tgz>.
- [2] M. S. Agranovich. Theorem on matrices depending on parameters and its applications to hyperbolic systems. *Functional Analysis and Its Applications*, 6:85–93, 1972.
- [3] Robert Beig. Concepts of hyperbolicity and relativistic continuum mechanics. *Lect. Notes Phys.*, 692:101–116, 2006.
- [4] Horst Beyer and Olivier Sarbach. On the well posedness of the Baumgarte-Shapiro-Shibata-Nakamura formulation of Einstein’s field equations. *Phys. Rev. D*, 70:104004, 2004.
- [5] Horst R. Beyer. Beyond partial differential equations. LNM 1898, 2005.
- [6] Gioel Calabrese, Ian Hinder, and Sascha Husa. Numerical stability for finite difference approximations of Einstein’s equations. *J. Comp. Phys.*, 218:607–634, 2005.
- [7] D. Christodolou. *The Action Principle and Partial Differential Equations*. Annals of Mathematics Studies, 146. Princeton University Press, 2000.
- [8] Robert P. Geroch. Partial differential equations of physics. In G. Hall, editor, *General Relativity*. 1996.
- [9] C. Gundlach and J. M. Martín-García. Symmetric hyperbolicity and consistent boundary conditions for second-order Einstein equations. *Phys. Rev. D*, 70:044032, 2004. gr-qc/0403019.
- [10] C. Gundlach and J.M. Martín-García. Symmetric hyperbolic form of systems of

- second-order evolution equations subject to constraints. *Phys. Rev. D*, 70:044031, 2004.
- [11] Carsten Gundlach and Jose M. Martín-García. Hyperbolicity of second-order in space systems of evolution equations. *Class. Quantum Grav.*, 23:S387–S404, 2006.
 - [12] Carsten Gundlach and Jose M. Martin-Garcia. Well-posedness of formulations of the Einstein equations with dynamical lapse and shift conditions. *Phys. Rev. D*, 74:024016, 2006.
 - [13] Bertil Gustafsson. *High Order Difference Methods for Time Dependent PDE*. Springer-Verlag, Berlin, Heidelberg, 2008.
 - [14] Bertil Gustafsson, Heinz-Otto Kreiss, and Joseph Oliger. *Time dependent problems and difference methods*. Wiley, New York, 1995.
 - [15] L. E. Kidder, Mark A. Scheel, and Saul A. Teukolsky. Extending the lifetime of 3D black hole computations with a new hyperbolic system of evolution equations. *Phys. Rev. D*, 64:064017, 2001.
 - [16] H.-O. Kreiss, O. E. Ortiz, and N. A. Petersson. Initial-boundary value problems for second order systems of partial differential equations. *ArXiv e-prints*, December 2010.
 - [17] Heinz-Otto Kreiss. Initial boundary value problems for hyperbolic systems. *Comm. Pure Appl. Math.*, 23:277–298, 1970.
 - [18] Heinz Otto Kreiss and J. Lorenz. *Initial-boundary value problems and the Navier-Stokes equations*. Academic Press, New York, 1989.
 - [19] Heinz-Otto Kreiss, N. Anders Petersson, and Jacob Yström. Difference approximations for the second order wave equation. *SIAM J. Numer. Anal.*, 40:1940–1967, 2002.
 - [20] Lee Lindblom, Mark A. Scheel, Lawrence E. Kidder, Robert Owen, and Oliver Rinne. A new generalized harmonic evolution system. *Class. Quantum Grav.*, 23:S447–S462, 2006.
 - [21] Jose M. Martin-Garcia. xAct: tensor computer algebra. <http://www.xact.es/>.
 - [22] Guy Métivier. The block structure condition for symmetric hyperbolic systems. *Bulletin of the London Mathematical Society*, 32:689–702, 2000.
 - [23] G. Nagy, O. E. Ortiz, and O. A. Reula. Strongly hyperbolic second order Einstein’s evolution equations. *Phys. Rev. D*, 70:044012, 2004.
 - [24] O. Sarbach, G. Calabrese, J. Pullin, and M. Tiglio. Hyperbolicity of the BSSN system of Einstein evolution equations. *Phys. Rev. D*, 66:064002, 2002.
 - [25] Olivier Sarbach and Manuel Tiglio. Continuum and discrete initial-boundary value problems and einstein’s field equations. *Living Reviews in Relativity*, 15(9), 2012.
 - [26] Michael E. Taylor. *Pseudodifferential operators / Michael E. Taylor*. Princeton University Press, Princeton, N.J. :, 1981.
 - [27] Robert M. Wald. *General Relativity*. University of Chicago Press, Chicago, 1984.