Numerical Relativity
in higher dimensional spacetimes

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From August

Institute Superior Técnico
Outline of the talk

• Introduction

• Numerical Relativity in higher dimensional spacetimes
  – Decomposition of the Einstein equation
  – BSSN formalism
  – Gauge equations

• Black hole collision

• Black hole in AdS spacetimes

• Summary
Introduction
Verification for higher dimensional gravity

- In principle, BH can be formed when the energy is confined to the sufficient small region. (Cf. “Hoop-Conjecture” Thorne(1972))

- Planck scale is $10^{19}[\text{GeV}]$ if $D = 4$.

- Higher dimensional theory, Arkani-Hamed+(98), Randall+(99)

- Current experiments show the inverse square law of the gravity is valid up to $0.1 \sim 0.01 \text{mm}$.

- Planck scale could be larger than $10[\text{TeV}]$.

- Possibility of the BH formation for particle collisions
  Dimopoulos and Landsberg(2001), Giddings and Thomas(2002)
High-velocity Collisions in Higher Dimensions

It is believed that Gravity is dominant below the Planck scale. We could treat it as classical gravity.

- Test particle collisions
- High-velocity BH collisions  Shibata+(2008), Sperhake+(2009)

Impact paramter (BH formation)
M and J (feature of BH)
Dissipation (feature of GWs)

NR in Higher Dimensions
The evolution of Binary BHs is calculated quite accurately.

It’s also popular to study about the Gravitational Recoil (Kick).
High-velocity BH Collisions in Numerical Relativity

Orbits of a BH in 4D collision

Waveform of 5D Headon collision

- Sperhake, Cardoso, Pretorius, Berti, Hinderer, Yunes (2009)
- Shibata, HO, Yamamoto (2008)
- Witek, Zilhão, Gualtieri, Cardoso, Herdeiro, Nerozzi, Sperhake (2010)

Spin of the formed BH in 4D collision

Shibata, HO, Yamamoto (2008)

Witek, Zilhão, Gualtieri, Cardoso, Herdeiro, Nerozzi, Sperhake (2010)
Numerical Relativity
D-dimensional NR (D-1+1 decomposition)

To solve the Einstein equation

\[ \mathcal{R}_{ab} - \frac{1}{2} g_{ab} \mathcal{R} = \kappa T_{ab} (= 0). \]

**Spatial metric**

\[ \gamma_{ab} \equiv g_{ab} + n_a n_b \]

\( n^a \): Normal vector to the space \( (n^a = (1/\alpha, \beta^i/\alpha)) \)

**Extrinsic curvature**

\[ K_{ab} \equiv -\frac{1}{2\alpha} (\partial_t \gamma_{ab} - D_b \beta_a - D_a \beta_b) \]

**Decomposition of the Einstein equation**

\( \rightarrow \) **Same as 4D**

**Evolution equation**

\[ \partial_t K_{ab} = -D_b D_a \alpha + \alpha (R_{ab} - 2K_{ac} K^c_b + K_{ab} K^c_c) \]

\[ + \beta^c D_c K_{ab} + K_{cb} D_a \beta^c + K_{ca} D_b \beta^c \]

**Constraints**

**Hamiltonian constraint**

\[ R + K^2 - K_{ab} K^{ab} = 0 \]

**Momentum constraints**

\[ D_b K^b_a - D_a K^c_c = 0 \]
We cannot evolve $\gamma_{ij}$ and $K_{ij}$ numerically because of the existence of violating modes.

\[ \gamma_{ij} = \chi^{-1} \tilde{\gamma}_{ij}, \quad \det \tilde{\gamma}_{ij} = 1, \]

\[ K_{ij} = \chi^{-1} \tilde{A}_{ij} + \frac{1}{D-2} \chi^{-1} \gamma_{ij} K, \]

\[ \tilde{\Gamma}^i = -\tilde{\gamma}^{ij}_{,j}. \]

\[
\left( \partial_t - \beta^i \partial_i \right) \chi = \frac{2}{D-1} \chi \left( \alpha K - \partial_i \beta^i \right),
\]

\[
\left( \partial_t - \beta^l \partial_l \right) \tilde{\gamma}_{ij} = -2 \alpha \tilde{A}_{ij} + \tilde{\gamma}_{il} \partial_j \beta^l + \tilde{\gamma}_{jl} \partial_i \beta^l - \frac{2}{D-1} \tilde{\gamma}_{ij} \partial_l \beta^l,
\]

\[
\left( \partial_t - \beta^i \partial_i \right) K = \alpha \left[ \tilde{A}_{ij} \tilde{A}^{ij} + \frac{1}{D-1} K^2 \right] - D^i D_i \alpha,
\]

\[
\left( \partial_t - \beta^l \partial_l \right) \tilde{A}_{ij} = \chi \left[ \alpha \left( \tilde{R}_{ij} - \frac{1}{D-1} \gamma_{ij} \tilde{R} \right) - \left( D_i D_j \alpha - \frac{1}{D-1} \gamma_{ij} D^l D_l \alpha \right) \right]
\]

\[
+ \alpha \left( K \tilde{A}_{ij} - 2 \tilde{A}_{il} \tilde{A}^{lj} \right) + \tilde{A}_{lj} \partial_i \beta^l + \tilde{A}_{il} \partial_j \beta^l - \frac{D-2}{D-1} \tilde{A}_{ij} \partial_l \beta^l,
\]

\[
\left( \partial_t - \beta^j \partial_j \right) \tilde{\Gamma}^i = \tilde{\gamma}^{ik} \partial_j \partial_k \beta^i + \frac{D-3}{D-1} \tilde{\gamma}^{ij} \partial_j \partial_k \beta^k - \tilde{\gamma}^{ij} \partial_j \beta^i + \frac{2}{D-1} \tilde{\Gamma}^i \partial_j \beta^j
\]

\[
- 2 \tilde{A}^{ij} \partial_j \alpha + 2 \alpha \left( \tilde{\Gamma}^i_{jk} \tilde{A}^{jk} - \frac{D-1}{2} \partial_j \chi \tilde{A}^{ij} - \frac{D-2}{D-1} \tilde{\gamma}^{ij} \partial_j K \right).
\]
Gauge conditions

Puncture Gauge Conditions  Alcubierre+ (2003)

\[(\partial_t - \beta^i \partial_i) \alpha = -\eta_\alpha \alpha K,\]
\[(\partial_t - \beta^j \partial_j) \beta^i = -\eta_\beta B^i,\]
\[(\partial_t - \beta^j \partial_j) B^i = (\partial_t - \beta^j \partial_j) \tilde{\Gamma}^i - \eta_B B^i.\]

\(\eta_\alpha, \eta_\beta\) and \(\eta_B\) are arbitrary parameters. (We choose the parameter carefully by problems to solve.)

- we can treat the evolution near the singularity well.
- The result of the evolution of the Schwarzschild-Tangherlini BH:
  In the puncture gauge, there is a special slice as the attractor.

Dennison, Wendell, Baumgarte (2010)
SACRA code

Yamamoto, Shibata, Taniguchi (2008)

- How to solve the Einstein eq.
- Economical:
  - Space: 4th order finite differences
  - Time: 4th order Runge-Kutta integration

- BH-BH, NS-NS and BH-NS:
  To resolve following scales,
  - Size of the star,
  - Interval between stars,
  - Wavelength of GWs,
  we need Adaptive Mesh Refinement (AMR).

We can run on a computer,
  - CPU: 3.4 GHz, 6 cores,
  - MEM: 32 GB,
  - EURO: 1,300 euros.
5D Numerical Relativity (Cartoon method)

Cartoon Method  Alcubierre+(1999)

- Originally this is used to solve the axisymmetric problems.

- Numerical errors tend to grow near the axis ($r \sim 0$) on polar coordinates. (Because $\frac{1}{r}$ diverges apparently at $r = 0$.)

- There isn't such a problem on Cartesian coordinates.

We can get any values by the interpolation if we have the data on $z = 0$.

We can take Cartesian coordinates and save the costs by Cartoon method.
Black Hole Collision
**Initial condition (a Boost BH)**

**Schwarzschild-Tangherlini Black Hole in isotropic coordinates**

\[ \text{ds}^2 = -\alpha_0^2 \text{dt}_0^2 + \psi_0^{\frac{4}{D-3}} \left( \text{dw}_0^2 + \text{dx}_0^2 + \sum_{i=1}^{D-3} \text{dy}_0^2_i \right) \]

\[ \alpha_0 = \left( 1 - \frac{\mu}{4r_0^{D-3}} \right) \psi_0^{-1}, \quad \psi_0 = 1 + \frac{\mu}{4r_0^{D-3}}, \quad r_0^2 = w_0^2 + x_0^2 + \sum_{i=1}^{D-3} y_0^2_i \]

**a Boost BH solution**

\[ \text{ds}^2 = -\gamma^2 \left( \alpha_0^2 - v^2 \psi_0^{\frac{4}{D-3}} \right) \text{dt}^2 - 2\gamma^2 v \left( \psi_0^{\frac{4}{D-3}} - \alpha_0^2 \right) \text{dt} \text{dw} + \psi_0^{\frac{4}{D-3}} \left[ B_0^2 \text{dw}^2 + \text{dx}^2 + \sum_{i=1}^{D-3} \text{dy}_i^2 \right] \]

**Extrinsic curvature**

\[ K_{xx} = K_{yi_i} = \frac{2}{D-3} \frac{\gamma v \alpha_0 \psi_0' w}{\psi_0 B_0 r_0}, \]

\[ K_{ww} = \frac{\gamma^3 v B_0 w}{r_0} \left[ 2\alpha_0 - B_0^{-2} \left( \frac{2}{D-3} \alpha_0 \psi_0^{-1} \psi_0' - \alpha_0^2 \psi_0^{\frac{4}{D-3}} \alpha_0' \right) \right], \]

\[ K_{wx} = \frac{\gamma v B_0 x}{r_0} \left[ \alpha_0 - B_0^{-2} \left( \frac{2}{D-3} \alpha_0 \psi_0^{-1} \psi_0' - \alpha_0^2 \psi_0^{\frac{4}{D-3}} \alpha_0' \right) \right], \]

\[ K_{wy_i} = \frac{\gamma v B_0 y_i}{r_0} \left[ \alpha_0 - B_0^{-2} \left( \frac{2}{D-3} \alpha_0 \psi_0^{-1} \psi_0' - \alpha_0^2 \psi_0^{\frac{4}{D-3}} \alpha_0' \right) \right]. \]

Then, we consider the initial data for BHs with initial velocity by using Boost BHs.
Initial condition (Superposition of Boost BHs)

Initial condition for two Boost BHs

\[ \text{d} \ell^2 = \psi^{\frac{4}{D-3}} \left( B^2 \text{d}w^2 + \text{d}x^2 + \sum_{i=1}^{D-3} \text{d}y_i^2 \right) \]

**conformal factor**

\[ \psi = \psi_{\text{main}} + \delta \psi, \quad r_A = \sqrt{\gamma^2 (w - w_A)^2 + (x - x_A)^2 + \sum_{i=1}^{D-3} y_i^2} \quad (A = 1, 2) \]

\[ \psi_{\text{main}} \equiv 1 + \frac{\mu_1}{4r_1^{D-2}} + \frac{\mu_2}{4r_2^{D-2}} \]

\[ B^2 = \gamma^2 \left[ 1 - v^2 \psi_{\text{main}}^{-\frac{4}{D-3}} \left( 1 - \frac{\mu_1}{4r_1^{D-3}} - \frac{\mu_2}{4r_2^{D-3}} \right) \right] \]

We should get \( \delta \psi \) by solving the Hamiltonian constraint.

**extrinsic curvature**

\[ K_{ij} = K_{ij}^{(1)} + K_{ij}^{(2)} + \delta K_{ij} \]

\( \delta \psi \) and \( \delta K_{ij} \) are the small corrections when the BHs are sufficiently apart from each other.

We should also get \( \delta K_{ij} \) by solving the Momentum constraints.
Orbits of BH Collisions in 4D and 5D

**4D** \( v = 0.6 \), Zoom-whirl behavior, impact parameter \( (x = 5.0, 5.2, 5.4) \)

\[\begin{align*}
\text{4D} & \quad v = 0.6, \text{ Zoom-whirl behavior, impact parameter} (x = 5.0, 5.2, 5.4) \\
\end{align*}\]

**5D** \( v = 0.6 \), No Zoom-whirl, impact parameter \( (x = 1.65, 1.66, 1.75, 2.0) \)

\[\begin{align*}
\text{5D} & \quad v = 0.6, \text{ No Zoom-whirl, impact parameter} (x = 1.65, 1.66, 1.75, 2.0) \\
\end{align*}\]
Scattering of BHs

Vertical axis denotes **impact parameter**. Horizontal axis denotes **initial velocity of BHs**.

- $b_C$: Lower limit of $b$ which we can see the scattering.
- $b_B$: Upper limit of $b$ which we can see the merging.
Exchange of M and J by Scattering

- Mass and Angular momentum grow after scattering.
- “Tidal Heating effect” could explain this.
  (Cf. The membrane paradigm(1986), Poisson(2009,2010))
Kretschmann Invariant Scalar

Let's see the BH scattering in terms of the Gauge invariant quantity.

\[ \mathcal{K}^2 = R_{abcd}R^{abcd} \]

(Kretschmann invariant scalar)

Normalized by the value on the Horizon of SBH in 5D isotropic coordinates

\[ \mathcal{K}^2 = (D - 2)^2 (D - 1) (D - 3) E_P^4 \]

\[ E_P: \text{Planck Energy} \]

\[ \mathcal{K}^2 = 72 E_P^4 \]
Kretschmann Scalar during scattering

- Kretschmann invariant $\mathcal{K}$ on $y = z = 0$ plane.
- Red circles denote the shape of AH.
- Two BHs always exist through the scattering.
- Kretschmann becomes large outside the Horizon.

HO, Nakao, Shibata (2011)
• Horizontal axis denotes separation between two BHs.
  (Time goes from left to right.)

• Vertical axis denotes Kretschmann scalar at the center of mass.
Kretschmann vs Initial velocity of BH

Kretschmann invariant at the same impact parameter
Horizontal axis denotes the initial velocity of BHs.
Vertical axis denotes the Maximum of $K$ at the center of mass (log scale).
Black Hole in AdS spacetimes
BH in the spacetimes with a brane

**Looking back at the history briefly,**

L. Randall and R. Sundrum (1999)  
Warped spacetimes

BHs larger than bulk scale are not static.  
Method to make the solution with a BH. (Small BH could be static.)

It is difficult to make a large BH.

N. Tanahashi and T. Tanaka (2008)  
A larger BH seems unstable.

P. Figueras and T. Wiseman (2011)  
Method to make a large BH.

**Future Prediction?**  
**We want to see** Dynamical Evolution.
Exact Solution of Thick Brane

T. Takahashi, HO, M. Shibata (2012-13?)

(My) Requirement

- I want to start from SACRA code.
- I like easy boundary conditions.

We can make the static thick brane with a scalar field. (Cf. Giovannini (2002))

\[ S = \int d^5x \sqrt{-g} \left[ \frac{R}{2\kappa} - \frac{1}{2} \nabla_a \phi \nabla^a \phi - V(\phi) \right] \]  \hspace{1cm} (1)

\[ \phi(y) = \sqrt{\frac{3}{\kappa}} \arctan [\sinh(by)], \quad z = 2b \sinh(by) \]  \hspace{1cm} (2)

\[ V(\phi) = \frac{3b^2}{2\kappa} \left[ 1 - 5 \sin^2 \left( \sqrt{\frac{\kappa}{3}} \phi \right) \right] \]  \hspace{1cm} (3)

Metric of the Brane Solution \hspace{1cm} (Cf. RS II model)

\[ ds^2 = \frac{1}{1 + b^2 z^2} \left[ dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu \right] \]  \hspace{1cm} (4)

\[ ds^2 = \frac{\ell^2}{z^2 + \ell^2} \left[ dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu \right] \]

\[ b = \ell^{-1} \] is the thickness of the brane.
To solve **Einstein equation**

\[ \mathcal{R}_{ab} - \frac{1}{2} g_{ab} \mathcal{R} = \kappa T_{ab} \]

**Spatial metric**

\[ \gamma_{ab} \equiv g_{ab} + n_a n_b \]

\( n^a \): Normal vector \( (n^a = (1/\alpha, \beta^i/\alpha)) \)

**Extrinsic curvature**

\[ K_{ab} \equiv -\frac{1}{2\alpha} \left( \partial_t \gamma_{ab} - D_b \beta_a - D_a \beta_b \right) \]

Decomposition of the Einstein equation

\[ \rightarrow \text{Same form as usual} \]

**Evolution equation**

\[
\partial_t K_{ab} = \beta^c D_c K_{ab} + K_{cb} D_a \beta^c + K_{ca} D_b \beta^c \\
+ \alpha \left( R_{ab} - 2K_{ac} K^c_b + K_{ab} K - \kappa \left[ S_{ab} + \frac{\rho - S}{3} \gamma_{ab} \right] \right) \\
- D_b D_a \alpha
\]

**Constraints**

- **Hamiltonian constraint**
  \[ R + K^2 - K_{ab} K^{ab} = 2\kappa \rho \]

- **Momentum constraints**
  \[ D_b K^b_a - D_a K = \kappa j_a \]
BSSN formalism in AdS spacetimes

\[ ds^2 = \frac{1}{a^2(z)} \left[ -\alpha^2 dt^2 + \gamma_{ij} (dx^i + \beta^i dt)(dx^j + \beta^j dt) \right] \]  \hspace{1cm} (5)

- We define the BSSN variables without \( a(z) \) part.
- We get usual evolution equation with BSSN variables + \( a(z) \) terms.

\[
(\partial_t - \beta^i \partial_i) \chi = \frac{1}{2} \chi \left( \alpha K - \partial_i \beta^i \right) + \frac{2a'}{a} \beta^z \chi,
\]

\[
(\partial_t - \beta^l \partial_l) \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} + \tilde{\gamma}_{ii} \partial_j \beta^l + \tilde{\gamma}_{jl} \partial_i \beta^l - \frac{1}{2} \tilde{\gamma}_{ij} \partial_l \beta^l,
\]

\[
(\partial_t - \beta^i \partial_i) K = \alpha \left[ \tilde{A}_{ij} \tilde{A}^{ij} + \frac{1}{4} K^2 + \frac{\kappa}{3} (2\rho + S) \right] - \left[ a \tilde{D}_i \tilde{D}_i \left( \frac{\alpha}{a} \right) \right]^{TF} \frac{a'}{a} \beta^z K,
\]

\[
(\partial_t - \beta^l \partial_l) \tilde{A}_{ij} = \ldots,
\]

\[
(\partial_t - \beta^j \partial_j) \tilde{\Gamma}^i = \textbf{Evolution equations for the scalar field}
\]

\[
(\partial_t - \beta^l \partial_l) \phi = -\alpha \Pi,
\]

\[
(\partial_t - \beta^l \partial_l) \Pi = \alpha \kappa \Pi + \frac{\alpha}{a} \frac{\partial V}{\partial \phi} - \chi \alpha \left[ \tilde{D}^i \tilde{D}_i \phi + \left( \frac{\partial \alpha}{\alpha} - \frac{\partial \chi}{\chi} \frac{3a'}{a} \delta^z_i \right) \tilde{D}_i \phi \right].
\]
Coordinates and Boundary Conditions

**Coordinates**

- We want higher resolution near the brane.
- Perhaps outer region could be coarse. (not sure)
- We use \( y \) coordinate. (The interval of \( z \) is larger at far.)

\[
z = 2b \sinh(by)
\]

- But we defined BSSN variables on \( z(\gamma_{zz} \text{ conformal flat}) \).
- We should pay attention to the derivatives of \( z \).

\[
\partial_z \alpha = \frac{1}{a(y)} \partial_y \alpha
\]

**Boundary Conditions**

- \( Z_2 \) symmetry at the brane
  We can give the variables at \(-y\) from that at \( y \).

\[
\alpha(-y) = \alpha(y), \phi(-y) = -\phi(y)
\]

- If outer boundaries are enough far, we can impose the Neumann condition or Outgoing condition. (It could be correct unless we evolve for a long time.)
Hamiltonian Constraint

- We want an initial condition with Apparent Horizon (AH).
- They construct the solution with AH by introducing a scalar field as gravitational source.

Metric ansatz

\[ dl^2 = \frac{1}{a^2(z)} \left[ dz^2 + (1 + w(r, z))^4 \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right) \right] \]

Hamiltonian Constraint

\[
\partial_r^2 w + \frac{2}{r} \partial_r w + \frac{3}{2} \left[ \left( \partial_z^2 w - \frac{3 b z^2}{1 + b^2 z^2} \partial_z w \right) (1 + w)^4 + 3 (\partial_z w)^2 (1 + w)^3 \right] = -\pi (1 + w)^5 (\partial_z \psi)^2 - \pi (1 + w) (\partial_r \psi)^2
\]

We can solve the nonlinear elliptic equation (in principle if solutions exist)!
Initial Condition

How to give the scalar field:

\[ \psi(r, z) = \frac{f(r, z)}{(1 + w)^s} \]

\[ f(r, z) = \epsilon z \exp \left\{ -\frac{1}{2} \left( \frac{r^2}{a^2} + \frac{z^2}{b^2} \right) \right\} \]

We can choose \( s, a, b \) and \( \epsilon \). For example,

\[ a = b = 0.3, s = 2, \epsilon = 2.25 \]
• This is an evolution test for previous initial condition (Low res.).

• The values related to the AH are in about 4% error.

• It seems good before $t \sim 20$.

• The proper length of the equator is not so changed, but the position of AH on those coordinates is growing.
Summary

- We are in the stage to apply Numerical relativity to various spacetimes.
- We can see the dynamical evolution of the BH(s) in higher dimensions.
- Scattering of BHs in higher dimensions might be used to investigate the limit of classical gravity.
- The code of the AdS numerical relativity should show some interesting results (I hope).

Thank you very much for listening!