CHIRAL MODULATIONS IN CURVED SPACE

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based on*:

'Chiral Modulation in Curved Space I: Formalism', 2011
'CMCS II: Condensation in Singular Manifolds', to appear
'Chiral Phase Transitions Around Black Holes', 2011

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Outline

1. **Introduction**
2. **Four-Fermi Effective Theories**
3. **Effective Action and Heat-Trace**
4. High Densities
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6. **Black Holes**
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Introduction

– Strongly Coupled Theories have a very rich phase structure

• Collective Phenomena (Superconductivity; Condensation; Phase Transitions;...)
• Nuclear and Particle physics, Astrophysics and Cosmology, Condensed Matter, Gauge/Gravity correspondence

– QCD at Finite Temperature and Density

– Lattice vs Effective Field Theory Approach (NJL, GN, PNJL, QM, ...)

Fermion Effective Field Theories

Due to the difficulties to perform *ab initio* QCD simulations under generic external conditions (in particular at finite density), strongly coupled fermion effective field theories (Nambu-Jona Lasinio model; Gross-Neveu model) provide working models to describe dynamical chiral symmetry breaking in vacuum and hot/dense baryonic matter.

- share the global symmetries of QCD
- display the phenomena of χ_{SB}
- non-renormalizable
- Large-N and mean field approximations
- Allow (to some extent) analytical treatment

Chiral Symmetry Breaking

In flat space chiral symmetry is dynamically broken once the temperature drops below a critical temperature (in QCD, $T_c \sim 200$ MeV).

In the chirally broken phase, fermions acquire dynamically a mass and this is signalled by the appearance of a condensate.

When the density is low, the condensate assumes a constant value at which the free energy attains a minimum. This means that if chiral symmetry is broken (or restored) at a spacetime point, then it will be so everywhere.

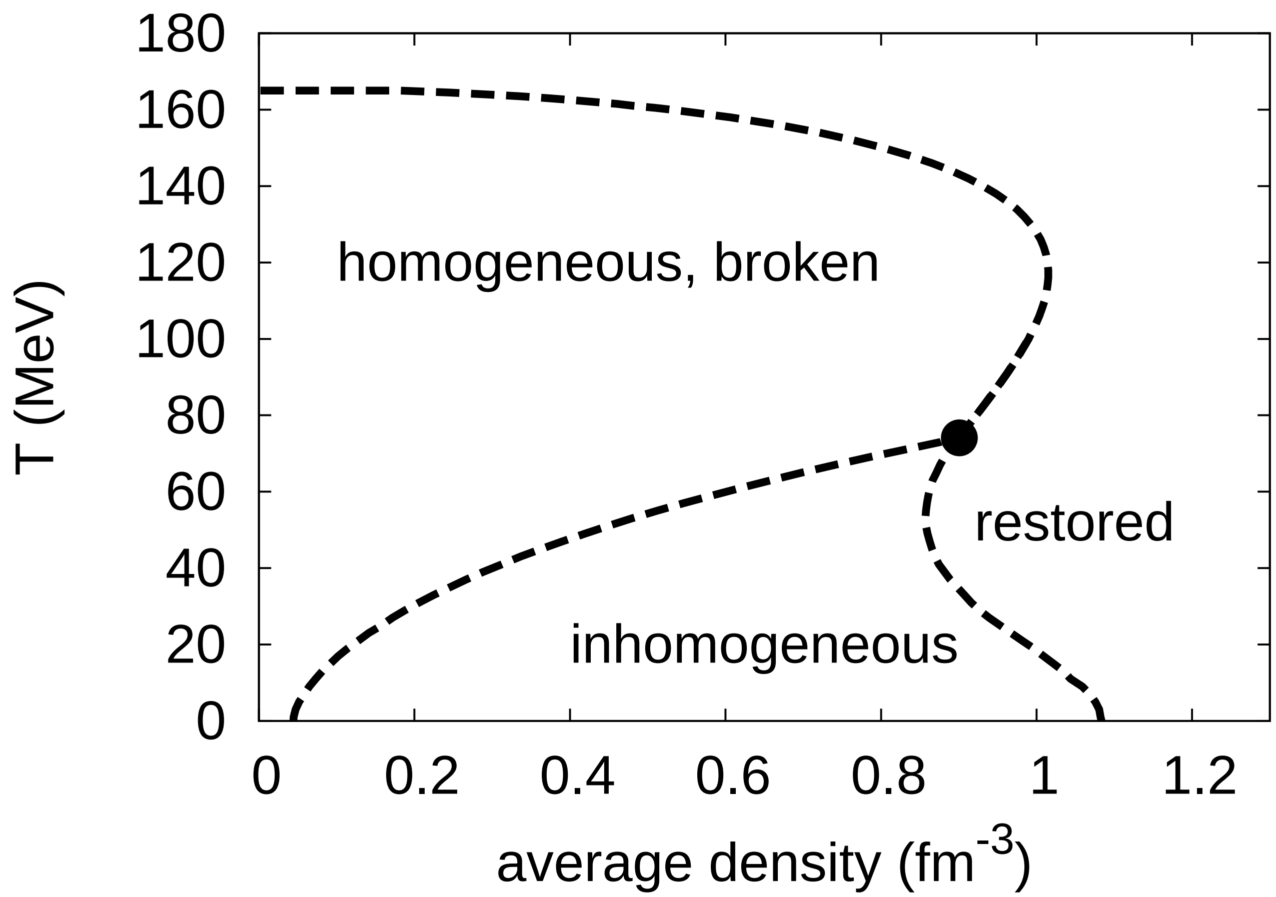
When density becomes large, the situation changes. The indication is that the condensate at high densities become inhomogeneous and rensembles a lattice of domain walls and chiral symmetry is broken inhomogeneously. This implies drastic changes in the phase structure of the theory.

The carachterization of the geography and morphology of the phase diagram of strongly coupled field theories is receiving a lot of attention

Chiral density wave approach; Ginzburg-Landau; in lower dimensional theories complete integrability is possible and inhomogeneous condensates can be constructed analytically. [Nakano; Tatsumi; Nickel; Buballa; Dunne; Basar; Thies; ...]

Phase diagram in the (P)NJL model

from Buballa, Carignano and Nickel, PRD 2011



Why Curved Space?

Fermion Effective Field Theories

Prototype model

$$S = \int d^D x \sqrt{g} \left\{ \bar{\psi} i \gamma^\mu \nabla_\mu \psi + \frac{G}{2N} \left(\bar{\psi} \psi \right)^2 + \cdots \right\} \ ,$$

- ψ is a $(D \times N_f \times N_c)$-component quark spinor,
- N_f flavors and N_c colors $(N \equiv N_f \times N_c)$,
- G is the coupling constant.
- The dots stand for terms with higher mass dimension.

invariant under discrete chiral transformations

If the chiral symmetry is broken dynamically, the composite operator $\langle \bar{\psi} \psi \rangle$ acquires a non-zero vev and a fermion mass term would appear.

Consider a $D = d+1$ dimensional, ultra-static spacetime

$$ds^2 = dt^2 + g_{ij} dx^i dx^j \ ,$$

g_{ij} is the metric on the spatial section $\mathscr{M}$

Allowing a mean field value $\langle \bar{\psi}\psi \rangle = -N\sigma(x)/G$ for the chiral condensate, after bosonization, the partition function can be expressed as a path integral over σ:

$$Z = \int [d\sigma] \, e^{i\mathscr{S}_{eff}}$$

The effective action, $\mathscr{S}_{eff}$, per fermionic degree of freedom at the lowest order in the large-N approximation can be written as

$$\mathscr{S}_{eff} \;=\; -\int d^D x \sqrt{g} \left(\frac{\sigma^2}{2G}\right) + \ln \mathrm{Det} \left(i\gamma^\mu \nabla_\mu - \sigma + \mu\gamma^0\right)$$

The above determinant acts on the Dirac spinor, and coordinate space. The last term, proportional to μ, represents a chemical potential term that allows to include finite density effects, while finite temperature is introduced using the imaginary time formalism, $t \to -i\tau$ with period $\beta = 2\pi/T$, and imposing anti-periodic boundary conditions on the fermion fields, $\psi(\tau) = -\psi(\tau + \beta)$.

Squaring the Dirac operator allows one to express the effective action as

$$\mathscr{S}_{eff} = -\int d^D x \sqrt{g} \left(\frac{\sigma^2}{2G}\right) + \frac{1}{2} \ln \mathrm{Det} \left[\Box + \frac{1}{4}R + \sigma^2 - \mu^2 - 2i\mu\frac{\partial}{\partial t} + i\gamma^\mu \left(\nabla_\mu \sigma\right)\right]$$

Making finite temperature effects explicit leads to

$$\mathscr{S}_{eff} = -\int d^D x \sqrt{g} \left(\frac{\sigma^2}{2G}\right) + \frac{1}{2}\sum_{\lambda}\sum_{n=-\infty}^{\infty} \ln \mathrm{Det}\, \mathscr{D}^{(n)}$$

where

$$\mathscr{D}^{(n)} \equiv -\Delta + \omega_n^2 + \frac{1}{4}R + \sigma^2 - \mu^2 - 2i\mu\omega_n + \lambda \left|\partial\sigma\right| \ ,$$

$$\omega_n = \frac{2\pi}{\beta}\left(n + \frac{1}{2}\right), \ \Delta = \frac{1}{\sqrt{g}}\partial_i \left(\sqrt{g} g^{ij} \partial_j\right), \ \lambda = \pm 1$$

Effective Action and Heat-Trace

$$\mathscr{S}_{eff} \;=\; -\int d^D x \sqrt{g} \left(\frac{\sigma^2}{2G} \right) + \delta\Gamma$$

$$\delta\Gamma = \frac{1}{2} \sum_{\lambda} \sum_{n=-\infty}^{\infty} \ln \mathrm{Det}\, \mathscr{D}^{(n)}$$

The Mellin transform of the heat-trace may be used to define a zeta function

$$\zeta(s) = \frac{1}{\Gamma(s)} \sum_{n,\lambda} \int_0^\infty dt\, t^{s-1} \mathrm{Tr}\, e^{-t\mathscr{D}^{(n)}}$$

and the effective action can be expressed in terms of the values of the zeta function and its derivative at $s = 0$ (understood as regularized by means of analytical continuation)

$$\delta\Gamma = \frac{1}{2} \int d^d x \sqrt{g} \left(\zeta(0) \ln \ell^2 + \zeta'(0) \right)$$

Ginzburg-Landau Expansion

If the Seeley-DeWitt expansion is used

$$\mathrm{Tr}\, e^{-t\mathscr{D}^{(n)}} = \frac{1}{(4\pi t)^{d/2}} \sum_{j=0}^{\infty} \mathscr{H}_j\, t^j$$

the GL expansion for the partition function is easily obtained. In flat space, this reads

$$\begin{aligned} \mathscr{S}_{GL} \;=\; & \frac{\alpha_2}{2}\sigma^2 + \frac{\alpha_4}{4}\left[\sigma^4 + (\nabla\sigma)^2\right] \\ & + \frac{\alpha_6}{6}\left[\sigma^6 + 5\,(\nabla\sigma)^2\,\sigma^2 + \frac{1}{2}\,(\Delta\sigma)^2\right] + \cdots \end{aligned}$$

$\mathscr{H}_j$ are the standard heat-kernel coefficients

Resummation

Adapting the ansatz of Jack, Parker and Toms,

$$\mathrm{Tr}\, e^{-t\mathscr{D}^{(n)}} = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-t\mathscr{Q}} \sum_k \mathscr{C}_\lambda^{(k)}\, t^k$$

with $\mathscr{Q} = \omega_n^2 + R/12 + \sigma^2 - \mu^2 - 2i\mu\omega_n + \lambda\,|\partial\sigma|$allows a partial resummation of the GL-expansion. In the case of a manifold without boundary:

$$\begin{aligned} \mathscr{C}_\lambda^{(0)} &= 1\ , \quad \mathscr{C}_\lambda^{(1)} = 0\ , \\ \mathscr{C}_\lambda^{(2)} &= \mathscr{R} + \frac{1}{6}\Delta\left(\sigma^2 + \lambda\,|\partial\sigma|\right)\ , \end{aligned}$$

where

$$\mathscr{R} = \frac{1}{180} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - \frac{1}{180} R_{\mu\nu} R^{\mu\nu} - \frac{1}{120}\Delta R\ .$$

Using the above machinery gives

$$\zeta(s) = \frac{1}{\Gamma(s)} \sum_{k,\lambda} \int_0^\infty dt \, \frac{t^{s-1+k}}{(4\pi t)^{\frac{d}{2}}} \mathscr{C}_\lambda^{(k)} e^{-t\mathscr{X}_\lambda} \mathscr{F}_{\beta,\mu}(t)$$

where we have defined

$$\begin{aligned} \mathscr{F}_{\beta,\mu}(t) &= \sum_{n=-\infty}^{\infty} e^{-t\left(\omega_n^2 - 2i\mu\omega_n - \mu^2\right)} \\ \mathscr{X}_\lambda &= \left(R/12 + \sigma^2 + \lambda\,|\partial\sigma|\right) \end{aligned}$$

$s \in \mathbb{C}$ works as a regulator. We are assuming that $\Re s < d/2 - k - 2$ and then proceed by analytical continuation to $s = 0$ at the end. Under these assumptions the above expression is well defined and convergent for $\mu = 0$. The case of non vanishing chemical potential can be addressed by modifying the process of analytical continuation.

The analytical continuation can be carried out explicitly:

$$
\begin{aligned}
\zeta(0) &= \frac{\beta}{(4\pi)^{D/2}} \sum_{\lambda} \sum_{k=0}^{[D/2]} \gamma_k(D) \mathscr{C}_{\lambda}^{(k)} \mathscr{X}_{\lambda}^{D/2-k} \\
\zeta'(0) &= \frac{\beta}{(4\pi)^{D/2}} \sum_{k=0}^{\infty} \sum_{\lambda} \Big(a_k(D) \mathscr{C}_{\lambda}^{(k)} \mathscr{X}_{\lambda}^{D/2-k} + \gamma_k(D) \mathscr{C}_{\lambda}^{(k)} \mathscr{X}_{\lambda}^{D/2-k} \ln \mathscr{X}_{\lambda} \\
&+ 2^{D/2+1-k} \mathscr{C}_{\lambda}^{(k)} (\mathscr{X}_{\lambda})^{D/4-k/2} \sum_{n=1}^{\infty} (-1)^n \frac{\cosh(\beta\mu n)}{(n\beta)^{D/2-k}} K_{k-D/2}\left(n\beta\sqrt{\mathscr{X}_{\lambda}}\right) \Big)
\end{aligned}
$$

The coefficients $\gamma_k(D)$ and $a_k(D)$ are given by

$$
\begin{aligned}
\gamma_k(D) &= \lim_{s\to 0} \frac{\Gamma(s+k-D/2)}{\Gamma(s)} , \\
a_k(D) &= \lim_{s\to 0} \frac{\Gamma(s+k-D/2)}{\Gamma(s)} \left(\psi^{(0)}(s+k-D/2) - \psi^{(0)}(s) \right) .
\end{aligned}
$$

CSB & Gravity

Understanding how gravity may affect CSB is certainly relevant to describe analogous transitions in cosmology and in astrophysics.

Attention has, so far, been limited to consider constant curvature spacetimes and homogeneous condensates. In the cases considered so far, once the chemical potential and the curvature are kept small, the non-trivial effect of the curvature is to shift homogeneously the value that the condensate would assume in flat space. Chiral symmetry would still be broken homogeneously and the ground state would not develop any energetically favoured inhomogeneous phase.

Hawking & Moss arguments

The question we would like to address here is what happens to the above picture when there is a strong and spatially varying gravitational field. Physically, we have in mind black holes in the early universe.

Hawking and Moss have considered a similar problem for the Higgs model of electroweak symmetry breaking.

Black holes radiate energy at a temperature inversely proportional to their mass.

As the black hole evaporates, its temperature rises, and at some point a bubble of a high temperature phase surrounding the horizon may form, if a phase transition occurs.

Hawking's indication was that, in the Higgs model, the associated high temperature phase would be too localized around the black hole, so that symmetry, effectively, would not be restored.

The same problem has been reconsidered by Moss taking into account the effect of trapped particles, *i.e.* particles emitted by the black hole and reflected back by the walls of the bubble. He indicated that, for some class of bag models, the picture may change and lead to a transient configurations of restored symmetry phase, localized around the horizon.

Intuitive Description

Consider a Schwarzschild black hole of mass m surrounded by strongly interacting fermions in thermal equilibrium. The asymptotic temperature given by $T_{BH} = (8\pi m)^{-1}$, and the local (Tolman) temperature is given by

$$T_{loc} = T_{BH}/\sqrt{f} \quad f = 1 - 2m/r$$

Asymptotically, chiral symmetry is restored when $T_{BH} > T_c$ while broken for $T_{BH} < T_c$. When $T_{BH} < T_c$, T_{loc} crosses the critical temperature at a certain radius. Within this radius, the symmetry will be restored. This indicates the possibility that a domain wall structure of the condensate surrounding the black hole will arise.

Effective Field theory approach

A quantitative analysis can be done as outlined before.

$$S = \int d^4x \sqrt{g} \left\{ \bar{\psi} i \gamma^\mu \nabla_\mu \psi + \frac{\lambda}{2N} \left(\bar{\psi}\psi \right)^2 \right\}$$

$$ds^2 = f dt^2 + f^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2)$$

$$\Gamma = -\int d^4x \sqrt{g} \left(\frac{\sigma^2}{2\lambda} \right) + \mathrm{Tr} \ln \left(i\gamma^\mu \nabla_\mu - \sigma \right)$$

$$\sigma \equiv -\frac{\lambda}{N} \bar{\psi}\psi$$

The non-ultrastaticity of black hole spacetimes complicates things a bit.

Perform a conformal rescaling of the metric

$$d\hat{s}^2 = f^{-1} ds^2$$

Evaluate the effective action in the conformally related spacetime, $\hat{\Gamma}$

Transform back to the original spacetime (add the cocycle function), $\delta\Gamma$

$$\Gamma \;=\; -\int d^4x\sqrt{g}\left(\frac{\sigma^2}{2\lambda}\right) + \hat{\Gamma} + \delta\Gamma$$

$$\hat{\Gamma} = \frac{1}{2}\sum_{\epsilon=\pm}\sum_{n=-\infty}^{\infty} \mathrm{Tr}\ln\left[-\hat{\Delta} + \omega_n^2 + \mathscr{A} + f\sigma_\epsilon^2\right],$$

$$\sigma_\epsilon^2 := \sigma^2 + \epsilon f^{1/2}\sigma'$$

$$\mathscr{A}^{(n)} = f\left((n-2)\Delta \ln f/4 - (n-2)^2(\nabla \ln f)^2/16\right)$$

$$\hat{\Gamma} = \frac{1}{2}\int d^3x\sqrt{\hat{g}}\left[\zeta(0)\ln \ell^2 + \zeta'(0)\right]$$

$$\zeta(s) := \frac{1}{\Gamma(s)}\sum_{n,\epsilon}\int dt t^{s-1}\mathrm{Tr}\ e^{-t\left(-\hat{\Delta}+\omega_n^2+\mathscr{A}+f\sigma_\epsilon^2\right)}.$$

$$
\begin{aligned}
\hat{\Gamma} &= \frac{\beta}{2(4\pi)^2}\sum_{\epsilon}\int d^3x\sqrt{\hat{g}}\Bigg\{\frac{3\sigma_\epsilon^4}{4}-\left(\frac{\sigma_\epsilon^4}{2}+a_\epsilon\right)\ln\left(\frac{f\sigma_\epsilon^2}{\ell^2}\right)\\
&+ 16\frac{\sigma_\epsilon^2}{f\beta^2}\varpi_2(f^{\frac{1}{2}}\sigma_\epsilon)+4a_\epsilon\varpi_0(f^{\frac{1}{2}}\sigma_\epsilon)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\varpi_\nu(u) &:= \sum_{n=1}^{\infty}(-1)^n n^{-\nu}K_\nu\left(n\beta u\right) \ ,\\
a_\epsilon &:= \frac{1}{180}\left(\hat{R}^2_{\mu\nu\tau\rho}-\hat{R}^2_{\mu\nu}-\hat{\Delta}\hat{R}\right)+\frac{1}{6}\hat{\Delta}\left(f\sigma_\epsilon^2\right) \ .
\end{aligned}
$$

The cocycle function can be expressed in terms of heat-kernel coefficients (Dowker, Moss)

$$\delta\Gamma = \lim_{n\to 4}\left(C_n^{(2)}[\hat{g}] - C_n^{(2)}[g]\right)/(n-4)\ .$$

$\mathscr{O} = \Box + V$, the part of the heat-kernel coefficient, relevant for our computation, is

$$C_n^{(2)}[g] = \frac{1}{(4\pi)^{\frac{n}{2}}}\frac{1}{2}\int d^n x\sqrt{g}\left(V^2 - \frac{1}{3}RV + \cdots\right),$$

where the dots represent terms that do not depend on V or disappear upon integration by parts. In the present case

$$\delta\Gamma \ = \ \frac{\beta}{2(4\pi)^2}\sum_{\epsilon=\pm}\int d^3x\sqrt{g}\left[\frac{\sigma_\epsilon^4}{2}\ln f - \frac{2\sigma_\epsilon^2}{f}\lim_{n\to 4}\frac{d\Lambda_n}{dn}\right]\ ,$$

where $\lim_{n\to 4} d\Lambda_n/dn = (f'^2 - 2ff'' + 4ff'/r)/24$.

Effective Equation for the Condensate

The problem is now reduced to finding extrema of the effective action Γ with respect to the condensate σ.

Ignoring fourth order derivatives of the condensate allows us to express the equation of motion for the condensate as a non-linear Schrödinger-like equation of the form

$$\sigma'' + \delta_1 \sigma' + \delta_2 \sigma'^2 + \mathscr{K} = 0$$

where the coefficients δ_i and $\mathscr{K}$ are functions of σ but independent of its derivatives.

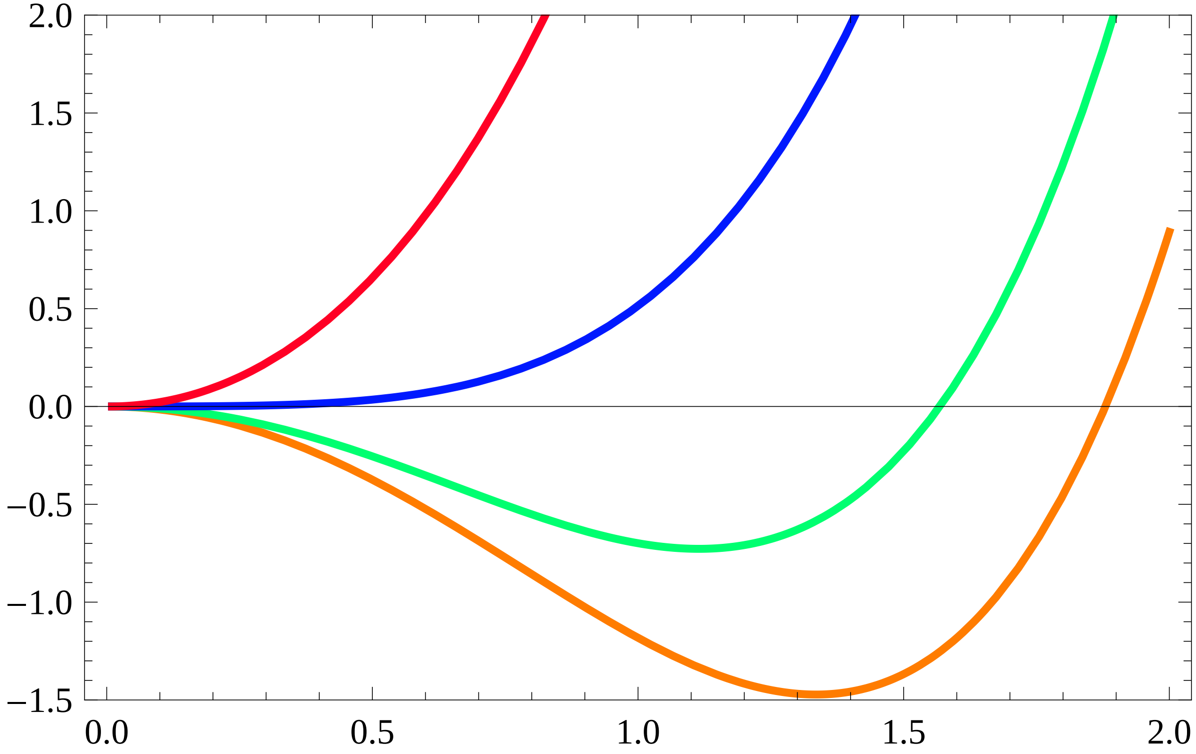
Asymptotics

In the asymptotic region, $r \to \infty$, we have

$$\partial_\sigma U_{as} = -\frac{3\sigma\left(4\lambda\sigma(4\varpi_{-1}(\sigma) + \beta\sigma \ln(\sigma/\ell) - 2\lambda\beta\sigma^2 + \beta\right)}{2\lambda\beta(-4\beta\sigma\varpi_1(\sigma) - 6\varpi_0(\sigma) + 3\ln(\sigma/\ell) - 2)}.$$

The critical temperature is determined by the equation $\partial_\sigma^2 U_{as}(\sigma) = 0$. Thus, expanding the Bessel functions contained in ϖ_ν for small σ, performing exactly the sum over n, and finally solving a trivial algebraic equation, one arrives at $T_c = \sqrt{3}\lambda^{-1/2}$.

Asymptotic Effective Potential



Local Effective Potential

Computing the thermodynamic potential locally will provide further insight on the form of the condensate. In fact, such a computation shows that starting from a set of parameters for which asymptotically the potential has a non vanishing minima, as we move towards the hole, the minima of the potential will gradually shift towards a configuration with vanishing σ.

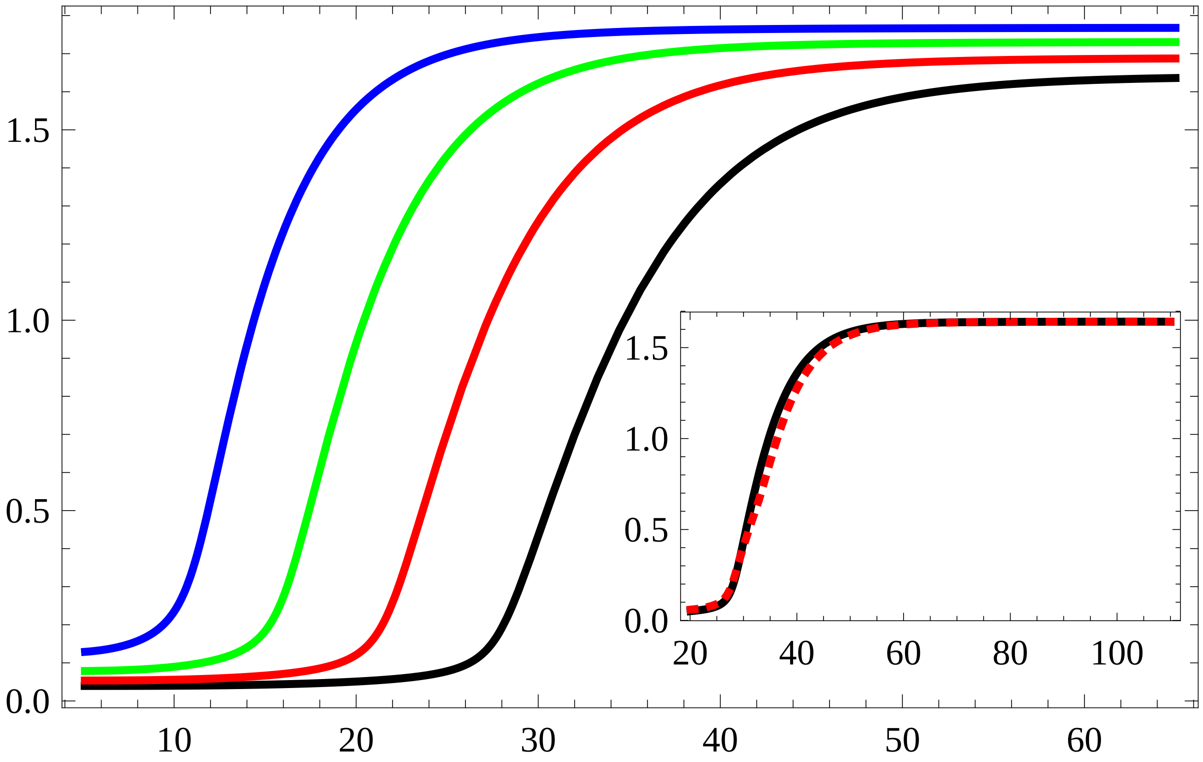
Numerics

Solving the effective equations for the condensate is not straightforward.

- Tuning of Boundary Conditions

- Infinite sums over Bessel functions and Matching

Inhomogeneous Solution



Inhomogeneous Solution

The solution has a kink structure: bubbles that separate a region of restored symmetry near the black hole from a region of broken symmetry surrounding it.

The size of the bubble is estimated by equating the local temperature to the critical temperature as

$$r_{bubble} \sim r_s / \left(1 - T_{BH}^2 / T_c^2\right)$$

Higher order corrections can be included systematically (computation done up to 4th order).

Higher order terms become less and less relevant as the black hole temperature gets closer to the critical one, due to the fact that the kink becomes increasingly thicker.

Discussion

– We considered a black hole immersed in a gas of strongly interacting fermions in approximate thermal equilibrium (primordial black holes; micro black holes). To analyze this problem, we have used the Hartle-Hawking vacuum state. The case of evaporating black holes requires, instead, the use of the Unruh vacuum. In that case, there is a net outgoing flux at infinity, the asymptotic energy density will decay like $1/r^2$, and the effective temperature vanishes asymptotically. In this case, the critical temperature is always larger than the asymptotic one, a bubble of restored symmetry is expected to form for black holes of any mass.

– Chromosphere formation around primordial black holes [Carr, Page & MacGibbons; Kapusta; Heckler; ...]

scattering occurs coherently in the radial direction
particles simply freely stream away to infinity, reducing their velocity due to gravitational attraction due to the black hole
Inside the bubble particles stay almost massless, scattering will occur only near the bubble wall, and inside the bubble processes that randomize the particle motion will not be important

Such a configuration won't be a chromosphere, rather more similar to the shock produced by stellar wind.

in progress

– What happens near a singularity (X)

– Charge (X) and Rotation

– Confinement/Deconfinement transitions and hadronization processes in black hole evaporation (in progress)

– higher dimensions