The Problem of a self gravitating scalar field with positive cosmological constant

Annales Henri Poincaré, arXiv:1206.4153 [gr-qc]

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Self-gravitating scalar field

The ”simplest”, non-pathological matter model with dynamical degrees of freedom in spherical symmetry:

- Electro-vacuum has no dynamical degrees of freedom - Birkhoff’s theorem completely determines its local structure.
- Dust is deemed pathological - it is known to develop singularities even in the absence of gravity, i.e. in a fixed Minkowski background.

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This conjecture as been verified for a variety of matter models and/or symmetry conditions:

Wald ('83), Friedrich ('86), Rendall and Tchapnda ('03), Rendall ('04), Anderson ('05), Ringström ('08), Rodnianski and Speck ('09), Beyer ('09), Valiente Kroon and Lübe ('11).

de Sitter space-time can be defined as the 4-dimensional hyperboloid

$$-x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = H^{-2}$$

in 5-dimensional Minkowski space.

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- It's a solution of the vacuum Einstein equations with positive cosmological constant $\Lambda = 3H^2$,

$$R_{\mu\nu} = \Lambda g_{\mu\nu}$$

- It's a space of maximal symmetry and of constant positive scalar curvature.

- There exist global coordinates $(\hat{t}, \chi, \theta, \varphi) \in \mathbb{R} \times \mathbb{S}^3$

$$ds^2 = -d\hat{t}^2 + H^{-2}\cosh(H\hat{t})\left(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)\right)$$

- There exist local coordinates $(t, x, y, z) \in \mathbb{R}^4$

$$ds^2 = -dt^2 + e^{2Ht}\left(dx^2 + dy^2 + dz^2\right)$$

This chart covers the region $x_0 + x_1 > 0$.

- The three FLRW possibilities can be realized on subsets of de Sitter.
- de Sitter with large Λ can be used to model inflation periods.
- de Sitter with small Λ can be used to model “recent” period of accelerated expansion.

Bondi spherical symmetry

$$\mathbf{g} = -g(u,r)\tilde{g}(u,r)du^2 - 2g(u,r)dudr + r^2 d\Omega^2$$

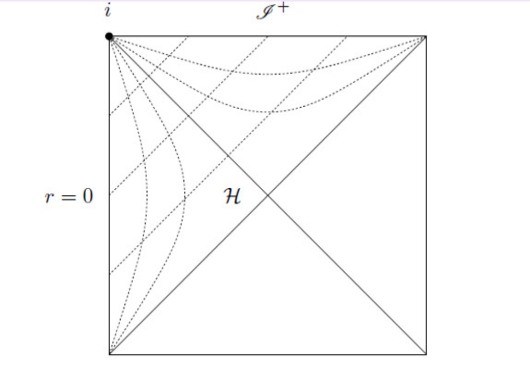
where $d\Omega^2$ is the round metric of the two-sphere, and

$$(u,r) \in [0,U) \times [0,R) \quad , \quad U,R \in \mathbb{R}^+ \cup \{+\infty\} \ .$$

- *radius function*, defined by $r(p) := \sqrt{\text{Area}(\mathcal{O}_p)/4\pi}$ (where $\mathcal{O}_p$ is the orbit through p)
- $u =$ *constant* are the future null cones of points at $r = 0$

Bondi coordinates (u, r, θ, φ) map the causal future of any point isometrically onto $\left([0, \infty) \times [0, \infty) \times S^2, g\right)$,

$$\mathring{\mathbf{g}} = -\left(1 - \frac{\Lambda}{3}r^2\right) du^2 - 2dudr + r^2 d\Omega^2$$



Main result

GLobal in time existence and uniqueness

Let $\Lambda > 0$ and $R > \sqrt{3/\Lambda}$.
Given small enough data, $\phi_0 \in \mathcal{C}^{k+1}([0, R])$, $k \geq 1$,

$$\sup_{0 \leq r \leq R} |\phi_0(r)| + \sup_{0 \leq r \leq R} |\partial_r \phi_0(r)| < \epsilon_0 \ ,$$

there exists a unique Bondi-spherically symmetric

$$\mathcal{C}^k([0, +\infty[\times [0, R] \times S^2)$$

solution $(M, \mathbf{g}, \phi)$ of the Einstein-Λ-scalar field system

$$R_{\mu\nu} = \kappa \, \partial_\mu \phi \, \partial_\nu \phi + \Lambda g_{\mu\nu}$$

with the scalar field ϕ satisfying the characteristic condition

$$\phi_{|u=0} = \phi_0 \ .$$

Moreover, we have the following bound in terms of initial data:

$$|\phi| \leq \sup_{0\leq r\leq R} |\partial_r (r\phi_0(r))| \ .$$

Regarding the asymptotics, there exists $\underline{\phi} \in \mathbb{R}$ such that

$$\left|\phi(u,r) - \underline{\phi}\right| \lesssim e^{-2Hu} \ ,$$

and

$$|\mathbf{g}_{\mu\nu} - \mathring{\mathbf{g}}_{\mu\nu}| \lesssim e^{-2Hu} \ ,$$

where $H := \sqrt{\Lambda/3}$ and $\mathring{\mathbf{g}}$ is de Sitter's metric in Bondi coordinates

The space-time is causally geodesically complete towards the future and has vanishing final Bondi mass.

A comment concerning regularity

- Regularity at the center does not require

$$\partial_r \phi_0(0) = 0$$

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  This follows from the wave equation!
- Instructive example: The solution of the spherically symmetric wave equation in Minkowski, with initial condition

$$\phi(r, r) = r$$

  is the smooth function

$$\phi(t, r) = t$$

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Einstein-Λ-scalar field
in Bondi coordinates

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In Bondi-spherical symmetry its full content is encoded in:

$$\frac{2}{r}\frac{1}{g}\frac{\partial g}{\partial r} = \kappa\,(\partial_r\phi)^2 \qquad (1)$$

$$\frac{\partial}{\partial r}(r\tilde{g}) = g\left(1 - \Lambda r^2\right) \qquad (2)$$

and the wave equation for the scalar field,

$$\nabla^\mu T_{\mu\nu} = 0 \Leftrightarrow \nabla^\mu\partial_\mu\phi = 0\ ,$$

which reads

$$\frac{1}{r}\left[\frac{\partial}{\partial u} - \frac{\tilde{g}}{2}\frac{\partial}{\partial r}\right]\frac{\partial}{\partial r}(r\phi) = \frac{1}{2}\left(\frac{\partial\tilde{g}}{\partial r}\right)\left(\frac{\partial\phi}{\partial r}\right) \qquad (3)$$

Consider the change of variable

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- In de Sitter wave equation becomes

$$Dh = -\frac{\Lambda}{3} r(h - \bar{h}) \quad , \quad D := \frac{\partial}{\partial u} - \frac{1}{2}\left(1 - \frac{\Lambda}{3} r^2\right)\frac{\partial}{\partial r}$$

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$$Dh = G\left(h - \bar{h}\right)$$

where

$$D = \frac{\partial}{\partial u} - \frac{\tilde{g}}{2}\frac{\partial}{\partial r} \quad \text{and} \quad G = \frac{1}{2r}\left[(g - \tilde{g}) - \Lambda g r^2\right]$$

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The scalar field is

$$\phi = \bar{h} := \frac{1}{r}\int_0^r h(u,s)ds$$

and, setting $g(u, r = 0) = 1$, the metric coefficients are given by

$$g(u,r) = \exp\left(\frac{\kappa}{2}\int_0^r \frac{\left(h - \bar{h}\right)^2}{s}ds\right) \quad , \quad \tilde{g}(u,r) = \bar{g} - \frac{\Lambda}{r}\int_0^r g s^2 ds$$

Let

$$\|h\|_{\mathcal{C}^0_{U,R}} := \sup_{u \leq U, r \leq R} |h(u,r)|$$

and define

$$\|h\|_{X_{U,R}} := \|h\|_{\mathcal{C}^0_{U,R}} + \|\partial_r h\|_{\mathcal{C}^0_{U,R}}$$

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If $\|h\|_X < \infty$

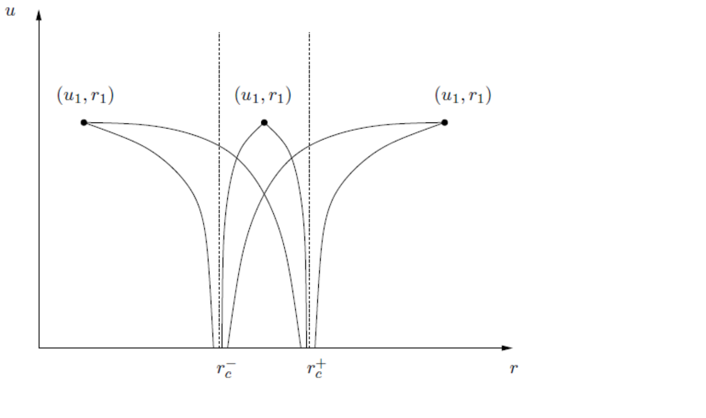
$$1 \le g \le e^{C\|h\|_X^2 R^2}$$

and if $\|h\|_X$ is small enough

$$G \le -C\, r \quad , \quad C = \frac{\Lambda}{3} + O(\|h\|_X)$$

These satisfy the ordinary differential equation,

$$\frac{dr}{du} = -\frac{1}{2}\tilde{g}(u,r).$$



Iteration

Consider the sequence defined by

$$\begin{cases} D_n h_{n+1} - G_n h_{n+1} = -G_n \bar{h}_n \\ h_{n+1}(0,r) = h_0(r) \end{cases}$$

integrate along the characteristics to obtain

$$h_{n+1}(u_1, r_1) = h_0(r_n(0)) e^{\int_0^{u_1} G_{n|\chi n} dv} - \int_0^{u_1} \left(G_n \bar{h}_n\right)_{|\chi n} e^{\int_u^{u_1} G_{n|\chi n} dv} du \, .$$

Goal: Show that $\|h_{n+1} - h_n\|_{X_{U,R}} \to 0$

Iteration

Recall that: $G \leq -C\,r$, for appropriately small $\|h\|_{X_{U,R}}$, then

$$
\begin{array}{rcl}
 & \cdots & \\
 & \Downarrow & \\
G_n & \leq & 0\,, \\
 & \Downarrow & \\
\|h_n\|_{C^0_{U,R}} & = & \|h_0\|_{C^0_R}\,, \\
 & \Downarrow & \\
\|h_n\|_{X_{U,R}} & \leq & (1+C^*)\|h_0\|_{X_R}\,, \\
 & \Downarrow & \\
G_{n+1} & \leq & 0\,, \\
 & \Downarrow & \\
 & \cdots &
\end{array}
$$

Fundamental fact: C^* does not depend on U!

- Solve local (in u) problem and obtain solution

$$h_1 : [0, U_1] \times [0, R] \to \mathbb{R} \quad , \quad U_1 = U(\Lambda, R, \|h_0\|_{X_R}) > 0$$

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- $\|h_1\|_{X_{U_1,R}} \leq (1 + C^*)\|h_0\|_{X_R}$

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- $\|h_1\|_{X_{U_1,R}} \leq (1 + C^*)\|h_0\|_{X_R}$
- Solve problem with initial data $h_1(U_1, \cdot)$ to obtain solution

$$h_2 : [0, U_2]\times[0, R] \to \mathbb{R} \quad , \quad U_2 = U(\Lambda, R, \|h_1(U_1, \cdot)\|_{X_R}) > 0$$

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- Extend original solution

$$h(u, r) := \begin{cases} h_1(u, r) & , \quad u \in [0, U_1] \\ h_2(u, r) & , \quad u \in [U_1, U_1 + U_2] \, . \end{cases}$$

- $\|h\|_{X_{U_1+U_2,R}} \leq (1 + C^*)\|h_0\|_{X_R}$
- We may extend the solution by the amount U_2 as before.

Exponential decay

Define

$$\mathcal{E}(u) := \|\partial_r h(u,\cdot)\|_{\mathcal{C}^0_R} \,.$$

The evolution equations and previous estimates imply that

$$\mathcal{E}(u_1) \leq \mathcal{E}(u_0) e^{-2C_1 \int_{u_0}^{u_1} r(s)ds} + C_2 \int_{u_0}^{u_1} \mathcal{E}(u) e^{-2C_1 \int_{u}^{u_1} r(v)dv} du \,,$$

and by Gronwall's Lemma

$$\mathcal{E}(u_1) \lesssim \mathcal{E}(u_0) e^{-C_3 u_1}$$

with

$$C_3 = 2\sqrt{\frac{\Lambda}{3}} + O(\|h\|_X)$$

What's next

For small data:

- Solve global in r problem.
- Generalize to non-linear scalar fields

For large data:

- Formation of cosmological black holes?

$$g_{\mu\nu} = (\text{Schwarzschild-de Sitter})_{\mu\nu} + O(e^{-2Hu})$$