# The Newman-Penrose formalism and its use in numerical relativity


Andrea Nerozzi

CENTRA
Instituto Superior Técnico, Lisbon


December 21, 2009

II Workshop on Black Holes, Lisbon

# Outline

# Wave extraction for the 3+1 approach

Two methodologies are being used:

- **Zerilli approach**
  It relies upon the angular properties of background vs radiation: the spherical background is subtracted and what remains are gravitational waves. It is well defined only for Schwarzschild background using a specific coordinate system.
- **Newman-Penrose formalism**
  For a specific tetrad choice, the Weyl scalar $\Psi_4$ is directly related to the gravitational wave degrees of freedom. This procedure is well defined also for Kerr background. The tetrad identification is fundamental.

# Weyl scalars and spin coefficients

- We introduce and tetrad of null vectors: two real ($\ell^\mu$ and $n^\mu$) and two complex conjugates ($m^\mu$ and $\bar{m}^\mu$).

$$
\begin{aligned}
k &= m^\mu \ell^\nu \nabla_\nu \ell_\mu, & \rho &= m^\mu \bar{m}^\nu \nabla_\nu \ell_\mu, & \epsilon &= 2^{-1} \cdot \ell^\nu \mathcal{T}_\nu, \\
\nu &= n^\mu n^\nu \nabla_\nu \bar{m}_\mu, & \mu &= n^\mu m^\nu \nabla_\nu \bar{m}_\mu, & \gamma &= 2^{-1} \cdot n^\nu \mathcal{T}_\nu, \\
\sigma &= m^\mu m^\nu \nabla_\nu \ell_\mu, & \tau &= m^\mu n^\nu \nabla_\nu \ell_\mu, & \alpha &= 2^{-1} \cdot \bar{m}^\nu \mathcal{T}_\nu, \\
\lambda &= n^\mu \bar{m}^\nu \nabla_\nu \bar{m}_\mu, & \pi &= n^\mu \ell^\nu \nabla_\nu \bar{m}_\mu, & \beta &= 2^{-1} \cdot m^\nu \mathcal{T}_\nu.
\end{aligned}
$$

- where $\mathcal{T}_\nu = n^\mu \nabla_\nu \ell_\mu + m^\mu \nabla_\nu \bar{m}_\mu$.
- The Weyl tensor is replaced by Weyl scalars:

$$
\begin{aligned}
\Psi_0 &= C_{abcd} \ell^a m^b \ell^c m^d, \\
\Psi_1 &= C_{abcd} \ell^a n^b \ell^c m^d, \\
\Psi_2 &= C_{abcd} \ell^a m^b \bar{m}^c n^d,
\end{aligned}
\qquad
\begin{aligned}
\Psi_3 &= C_{abcd} \ell^a n^b \bar{m}^c n^d, \\
\Psi_4 &= C_{abcd} n^a \bar{m}^b n^c \bar{m}^d.
\end{aligned}
$$

# The Kinnersley tetrad

- For a single black hole space-time it is possible to define the Kinnersley tetrad, which in Boyer-Lindquist coordinates reads

$$
\begin{aligned}
\ell^\mu &= \left[\left(r^2 + a^2\right)/\Delta, 1, 0, a/\Delta\right], \\
n^\mu &= \left[r^2 + a^2, -\Delta, 0, a\right]/\left(2\Sigma\right), \\
m^\mu &= \left[ia\sin\theta, 0, 1, i/\sin\theta\right]/\left[\sqrt{2}\left(r + ia\cos\theta\right)\right],
\end{aligned}
$$

  where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$.
- The Kinnersley tetrad has the following properties:
  - The $\ell^\mu$ and $n^\mu$ vectors coincide with the two repeated principal null directions. (This fixes type I and type II rotations).
  - The only non vanishing scalar is $\Psi_2 = \frac{M}{(r+ia\cos\theta)^3}$.
  - The spin coefficient $\epsilon$ vanishes (this fixes the spin-boost).
  - These three conditions fix the tetrad completely and assure that the null vector $\ell^\mu$ is geodesic and affinely parametrized.

# Perturbation theory in the Newman-Penrose formalism

- Using the Kinnersley tetrad it is possible to derive a perturbation equation (Teukolsky master equation) for the Weyl scalars $\Psi_0$ and $\Psi_4$.

$$\Psi_0 \approx \frac{e^{i\omega(t-r)}}{r^5} \quad \Psi_4 \approx \frac{e^{i\omega(t-r)}}{r} \quad \text{Outgoing waves}$$
$$\Psi_0 \approx \frac{e^{i\omega(t+r)}}{r} \quad \Psi_4 \approx \frac{e^{i\omega(t+r)}}{r^5} \quad \text{Ingoing waves}$$

- The values for the Weyl scalars are tetrad invariant at first order in perturbation theory, as a consequence of the fact that in the limit of Kerr all the scalars except $\Psi_2$ vanish.
- The optimal tetrad for wave extraction is a tetrad that naturally converges to the Kinnersley tetrad in the limit of Petrov type D.

# Our goal:

- Fix the tetrad such that it converges to the Kinnersley tetrad in the limit of Petrov type D space-time.
- Six degrees of freedom to fix, or in the Newman-Penrose formalism three complex parameters.
  - **Type I rotation and type II rotation**: We use the concept of transverse frames already developed (Nerozzi et al., Beetle et al.).
  - **Type III (spin-boost) rotation**: We introduce our new result which uses Bianchi and Ricci identities to understand the relation between the spin coefficients (in particular $\epsilon$) and the spin-boost parameter. Once this relation is known, we can impose $\epsilon = 0$ and obtain the corresponding condition on the spin-boost parameter.

# Fixing type I and type II rotations

- Transverse frames have $\Psi_1 = \Psi_3 = 0$.
- There are three different transverse frames.
- One of the three transverse frames converges to the Kinnersley frame (but the spin/boost parameter is not fixed).
- They are well defined for a Petrov type I space-time, which is the interesting case from a numerical point of view.
- Every frame has an infinite number of tetrads connected by a spin/boost transformation.

# How to find transverse frames

- Direct calculation of the rotation parameters (Campanelli et al.).
- However, in transverse frames, Weyl scalars have the simple expression

$$\Psi_0 = -\frac{i\mathcal{B}^{-2}}{2} \cdot \Psi_-,$$
$$\Psi_2 = -\frac{1}{2\sqrt{3}} \cdot \Psi_+,$$
$$\Psi_4 = -\frac{i\mathcal{B}^{2}}{2} \cdot \Psi_-,$$

where

$$\Psi_\pm = I^{\frac{1}{2}} \left( e^{\frac{2\pi i k}{3}} \Theta \pm e^{-\frac{2\pi i k}{3}} \Theta^{-1} \right),$$

$$P = \left[ J + \sqrt{J^2 - (I/3)^3} \right]^{\frac{1}{3}}.$$

- $\mathcal{B}$ is the spin-boost parameter $\mathcal{B} = \left(\frac{\Psi_4}{\Psi_0}\right)^{\frac{1}{4}}$.
- $k$ identifies the three different transverse frames $k = \{0, 1, 2\}$.
- $\Theta$ is given by $\Theta = \sqrt{3} P I^{-\frac{1}{2}}$ and in the limit of Petrov type D one has $\Theta \to 1$.
- The transverse frame with $k = 0$ is also a quasi-Kinnersley frame as $\Psi_- \to 0$ in this frame.

# Fixing the spin-boost parameter - the Bianchi identities

- In order to find a general expression for the spin coefficient $\epsilon$ in the limit of Petrov type D, we introduce the rescaled spin coefficients, and rewrite the Bianchi identities:

$$
\begin{aligned}
D\Psi_+ &= -\tilde{\lambda}\Psi_- + 3\rho\Psi_+, \\
D\Psi_- &= \tilde{\lambda}\Psi_+ - (4\tilde{\epsilon} - \rho)\,\Psi_-. \\
\Delta\Psi_+ &= \tilde{\sigma}\Psi_- - 3\mu\Psi_+, \\
\Delta\Psi_- &= -\tilde{\sigma}\Psi_+ + (4\tilde{\gamma} - \mu)\,\Psi_-.
\end{aligned}
$$

where $D = \ell^\mu \nabla_\mu$ and $\Delta = n^\mu \nabla_\mu$.

- The rescaled coefficients are now given by $\tilde{\lambda} = i\sqrt{3}\lambda\mathcal{B}^{-2}$, $\tilde{\sigma} = i\sqrt{3}\sigma\mathcal{B}^2$, $\tilde{\epsilon} = \epsilon + \frac{1}{2}D\ln\mathcal{B}$ and $\tilde{\gamma} = \gamma + \frac{1}{2}\Delta\ln\mathcal{B}$.

# Fixing the spin-boost parameter - the Ricci identities

- It is possible to show from one of the Ricci identities that $\tilde{\epsilon} = \ell^\mu \nabla_\mu \mathcal{H}$.
- Other Ricci identities can be used to find an equation for $\mathcal{H}$ in the limit of Petrov type D space-time, giving

$$\nabla^\mu \nabla_\mu \mathcal{H} + \nabla^\mu \ln\left(I^{\frac{1}{6}}\right) \nabla_\mu \left(2\mathcal{H} + \ln I^{\frac{1}{12}}\right) = -2\Psi_2.$$

- Using Brill-Lindquist coordinates, the solution for this equation can be found and is given by

$$\mathcal{H} = \frac{1}{2} \ln\left(\Gamma^{\frac{1}{2}} I^{\frac{1}{6}} \sin\theta\right).$$

  where $\Gamma = r^2 - 2Mr + a^2$.
- Finally, using the definition for the reduced $\tilde{\epsilon} = \epsilon + \frac{1}{2}\ell^\mu \nabla_\mu \ln \mathcal{B}$, we find the final expression for the spin coefficient $\epsilon$:

$$\epsilon = \frac{1}{2}\ell^\mu \nabla_\mu \ln\left(\Gamma^{\frac{1}{2}} I^{\frac{1}{6}} \mathcal{B}^{-1} \sin\theta\right).$$

# Summing up...

- The condition $\epsilon = 0$ gives the condition on the spin-boost parameter:

$$\mathcal{B} = \mathcal{B}_0 I^{\frac{1}{6}} \Gamma^{\frac{1}{2}},$$

where $\mathcal{B}_0$ is an integration constant.

- The expressions for the scalars are then given by

Ingoing $\ell^\mu$:

$$\begin{aligned} \Psi_0 &= \mathcal{B}_0^{-2} \cdot \Gamma^{-1} I^{\frac{1}{6}} \left(\Theta - \Theta^{-1}\right), \\ \Psi_2 &= -\frac{1}{2\sqrt{3}} \cdot I^{\frac{1}{2}} \left(\Theta + \Theta^{-1}\right), \\ \Psi_4 &= \mathcal{B}_0^{2} \cdot \Gamma I^{\frac{5}{6}} \left(\Theta - \Theta^{-1}\right). \end{aligned}$$

Outgoing $\ell^\mu$:

$$\begin{aligned} \Psi_0 &= \mathcal{B}_0^{-2} \cdot \Gamma I^{\frac{5}{6}} \left(\Theta - \Theta^{-1}\right), \\ \Psi_2 &= -\frac{1}{2\sqrt{3}} \cdot I^{\frac{1}{2}} \left(\Theta + \Theta^{-1}\right), \\ \Psi_4 &= \mathcal{B}_0^{2} \cdot \Gamma^{-1} I^{\frac{1}{6}} \left(\Theta - \Theta^{-1}\right). \end{aligned}$$

- The factor $\Gamma$ in the final expression depends on the black hole parameters and particular care must be taken in a numerical simulation for its calculation.

# Conclusions

- Using transverse frames and the Ricci identities, it is possible to fix all the three rotation paramaters, such that the tetrad naturally converges to the right Kinnersley tetrad in the limit of Petrov type D space-time.
- The final expressions for the scalars are functions of curvature invariants (which is not surprising as we have fixed all the gauge degrees of freedom).
- We aim to improve the final expression by relating the function $\Gamma$ and the integration constant $\mathcal{B}_0$ to invariant properties of the space-time.
- Next step: numerical implementation and comparison with other methodologies (Gram-Schmidt regularization to find the tetrad).