Solar System Tests of Hořava-Lifshitz Black Holes

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Abstract

Recently, a renormalizable gravity theory with higher spatial derivatives in four dimensions was proposed by Hořava. The theory reduces to Einstein gravity with a non-vanishing cosmological constant in IR, but it has improved UV behaviors. The spherically symmetric black hole solutions for an arbitrary cosmological constant, which represent the generalization of the standard Schwarzschild-(A)dS solution, has also been obtained for the Hořava-Lifshitz theory. The exact asymptotically flat Schwarzschild type solution of the gravitational field equations in Hořava gravity contains a quadratic increasing term, as well as the square root of a fourth order polynomial in the radial coordinate, and it depends on one arbitrary integration constant. The IR modified Hořava gravity seems to be consistent with the current observational data, but in order to test its viability more observational constraints are necessary. In the present paper we consider the possibility of observationally testing Hořava gravity at the scale of the Solar System, by considering the classical tests of general relativity (perihelion precession of the planet Mercury, deflection of light by the Sun and the radar echo delay) for the spherically symmetric black hole solution of Hořava-Lifshitz gravity. All these gravitational effects can be fully explained in the framework of the vacuum solution of Hořava gravity. Moreover, the study of the classical general relativistic tests also constrain the free parameter of the solution.

1 Introduction

1.1 General formalism

Hořava: renormalizable gravity theory in four dimensions which reduces to Einstein gravity with a non-vanishing cosmological constant in IR but with improved UV behaviors.

The latter theory admits a Lifshitz scale-invariance in time and space,

$$t \to l^z\, t \qquad x^i \to l\, x^i\,, \tag{1}$$

($z = 3$ for the case at hand), exhibiting a broken Lorentz symmetry at short scales, while at large distances higher derivative terms do not contribute, and the theory reduces to standard general relativity (GR).

Using the ADM formalism, the 4-dim metric is parameterized by the following

$$ds^2 = -N^2 c^2 dt^2 + g_{ij}\left(dx^i + N^i\, dt\right)\left(dx^j + N^j\, dt\right). \tag{2}$$

The **Einstein-Hilbert action** is given by

$$S = \frac{1}{16\pi G}\int d^4x\, \sqrt{g}\, N\left(K_{ij}K^{ij} - K^2 + R^{(3)} - 2\Lambda\right), \tag{3}$$

$R^{(3)}$ is the three-dimensional curvature scalar for g_{ij},
K_{ij} is the extrinsic curvature, defined as

$$K_{ij} = \frac{1}{2N}\left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right)\,. \tag{4}$$

The **IR-modified Hořava action** is given by

$$\begin{aligned} S \;=\; & \int dt\, d^3x\, \sqrt{g}\, N\Bigg[\frac{2}{\kappa^2}\left(K_{ij}K^{ij} - \lambda_g K^2\right) - \frac{\kappa^2}{2\nu_g^4}C_{ij}C^{ij} + \frac{\kappa^2\mu}{2\nu_g^2}\epsilon^{ijk}R^{(3)}_{il}\nabla_j R^{(3)l}{}_k \\ & -\frac{\kappa^2\mu^2}{8}R^{(3)}_{ij}R^{(3)ij} + \frac{\kappa^2\mu^2}{8(3\lambda_g-1)}\left(\frac{4\lambda_g-1}{4}(R^{(3)})^2 - \Lambda_W R^{(3)} + 3\Lambda_W^2\right) \\ & +\frac{\kappa^2\mu^2\omega}{8(3\lambda_g-1)}R^{(3)}\Bigg], \end{aligned} \tag{5}$$

where κ, λ_g, ν_g, μ, ω and Λ_W are constant parameters.
C^{ij} is the Cotton-York tensor, defined as

$$C^{ij} = \epsilon^{ikl}\nabla_k\left(R^{(3)j}{}_l - \frac{1}{4}R^{(3)}\delta^j_l\right). \tag{6}$$

Note that the last term in Eq. (5) represents a 'soft' violation of the 'detailed balance' condition, which modifies the IR behavior. This IR modification term, $\mu^4 R^{(3)}$, with an arbitrary cosmological constant, represent the analogs of the standard Schwarzschild-(A)dS solutions, which were absent in the original Hořava model.

The fundamental constants of the speed of light c, Newton's constant G, and the cosmological constant Λ are defined as

$$c^2 = \frac{\kappa^2\mu^2|\Lambda_W|}{8(3\lambda_g-1)^2} \quad G = \frac{\kappa^2 c^2}{16\pi(3\lambda_g-1)} \quad \Lambda = \frac{3}{2}\Lambda_W c^2. \tag{7}$$

1.2 Static and spherically symmetric black hole solutions

Consider a static, spherically symmetric solution with the metric ansatz:

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{8}$$

By substituting the metric into the action, the resulting reduced Lagrangian, after angular integration, is given by

$$\begin{aligned}\mathcal{L} \;=\; & \frac{\kappa^2\mu^2}{8(1-3\lambda_g)}e^{(\nu+\lambda)/2}\Bigg[(2\lambda_g-1)\frac{(e^{-\lambda}-1)^2}{r^2} + 2\lambda_g\frac{e^{-\lambda}-1}{r}(\lambda' e^{-\lambda}) + \frac{\lambda_g-1}{2}(\lambda' e^{-\lambda})^2 \\ & - 2(\omega-\Lambda_W)(1-e^{-\lambda}(1-r\lambda')) - 3\Lambda_W^2 r^2\Bigg]\ , \end{aligned} \tag{9}$$

The equations of motions are

$$\begin{aligned} (2\lambda_g-1)\frac{(e^{-\lambda}-1)^2}{r^2} - 2\lambda_g\frac{e^{-\lambda}-1}{r}(-\lambda' e^{-\lambda}) + \frac{\lambda_g-1}{2}(\lambda' e^{-\lambda})^2 \\ -2(\omega-\Lambda_W)[1-e^{-\lambda}(1-r\lambda')] - 3\Lambda_W^2 r^2 = 0\ , \\ \frac{(\nu'+\lambda')}{2}\left[(\lambda_g-1)(\lambda' e^{-\lambda}) - 2\lambda_g\frac{e^{-\lambda}-1}{r} + 2(\omega-\Lambda_W)r\right] \\ +(\lambda_g-1)\left[(-\lambda''+(\lambda)')e^{-\lambda} - \frac{2(e^{-\lambda}-1)}{r^2}\right] = 0 \end{aligned}$$

by varying the functions ν and λ, respectively.

Now, imposing $\lambda_g = 1$, which reduces to the Einstein-Hilbert action in the IR limit, one obtains the following solution of the vacuum field equations in Hořava gravity,

$$e^{\nu(r)} = e^{-\lambda(r)} = 1 + (\omega-\Lambda_W)r^2 - \sqrt{r[\omega(\omega-2\Lambda_W)r^3+\beta]}, \tag{10}$$

where β is an integration constant.

Throughout this work, we consider the Kehagias-Sfetsos (KS) asymptotically flat solution, i.e., $\beta = 4\omega M$ and $\Lambda_W = 0$:

$$e^{\nu(r)} = 1 + \omega r^2 - \omega r^2\sqrt{1+\frac{4M}{\omega r^3}}. \tag{11}$$

If the limit $4M/\omega r^3 \ll 1$, from Eq. (11) we obtain the standard Schwarzschild metric of general relativity, $e^{\nu(r)} = 1 - 2M/r$, which represents a "Post-Newtonian" approximation of the KS solution of the second order in the speed of light.

There are two event horizons at

$$r_\pm = M\left[1 \pm \sqrt{1-1/(2\omega M^2)}\right]. \tag{12}$$

To avoid a naked singularity at the origin, impose the condition

$$\omega M^2 \geq \frac{1}{2}. \tag{13}$$

Note that in the GR regime, i.e., $\omega M^2 \gg 1$, the outer horizon approaches the Schwarzschild horizon, $r_+ \simeq 2M$, and the inner horizon approaches the central singularity, $r_- \simeq 0$.

2 Solar system tests for Hořava-Lifshitz black holes

Three fundamental tests which provide observational evidence for GR and its generalizations:

(1) Precession of the perihelion of Mercury;

(2) Deflection of light by the Sun;

(3) Radar echo delay observations.

2.1 The perihelion precession of Mercury

The line element, given by Eq. (8), provides the following equation of motion for r

$$\dot{r}^2 + e^{-\lambda}\frac{L^2}{r^2} = e^{-\lambda}\left(E^2 e^{-\nu} - 1\right), \tag{14}$$

where

$$e^{\nu}\dot{t} = E = \text{constant}, \qquad r^2\dot{\phi} = L = \text{constant}. \tag{15}$$

Prescription:
(1) change of variable $r = 1/u$ and using $d/ds = Lu^2 d/d\phi$;
(2) formally representing $e^{-\lambda} = 1 - f(u)$;
(3) defining:

$$G(u) \equiv f(u)u^2 + \frac{E^2}{L^2}e^{-\nu-\lambda} - \frac{1}{L^2}e^{-\lambda}. \tag{16}$$

We finally arrive at:

$$\underline{\underline{\frac{d^2u}{d\phi^2} + u = F(u),}} \tag{17}$$

where

$$F(u) = \frac{1}{2}\frac{dG(u)}{du}. \tag{18}$$

A circular orbit $u = u_0$ is given by the root of the equation $u_0 = F(u_0)$. Any deviation $\delta = u - u_0$ from a circular orbit must satisfy the equation

$$\frac{d^2\delta}{d\phi^2} + \left[1 - \left(\frac{dF}{du}\right)_{u=u_0}\right]\delta = O\left(\delta^2\right), \tag{19}$$

which is obtained by substituting $u = u_0 + \delta$ into Eq. (17). Therefore, in the first order in δ, the trajectory is given by

$$\delta = \delta_0 \cos\left(\sqrt{1 - \left(\frac{dF}{du}\right)_{u=u_0}}\,\phi + \beta\right), \tag{20}$$

where δ_0 and β are constants of integration.

The variation of the orbital angle from one perihelion to the next is

$$\phi = \frac{2\pi}{\sqrt{1-\left(\frac{dF}{du}\right)_{u=u_0}}} = \frac{2\pi}{1-\sigma}. \tag{21}$$

$\sigma \equiv$perihelion advance, which represents the rate of advance of the perihelion.
As the planet advances ϕ radians in its orbit, its perihelion advances $\sigma\phi$ radians.
From Eq. (21), σ is given by $\sigma = 1 - \sqrt{1-(dF/du)_{u=u_0}}$, or, for small $(dF/du)_{u=u_0}$, by

$$\sigma = \frac{1}{2}\left(\frac{dF}{du}\right)_{u=u_0}. \tag{22}$$

For a complete rotation we have $\phi \approx 2\pi(1+\sigma)$, and the advance of the perihelion is

$$\delta\phi = \phi - 2\pi \approx 2\pi\sigma. \tag{23}$$

For the KS solution, the perihelion precession is given by:

$$\delta\phi = \pi \frac{3\sqrt{\omega_0}\left\{2\left(x_0^3+\omega_0\right)x_0^5 + b^2\left[2x_0^6 + \left(6\omega_0 - 4\sqrt{\omega_0\left(4x_0^3+\omega_0\right)}\right)x_0^3 + \omega_0^2 - \sqrt{\omega_0^3\left(4x_0^3+\omega_0\right)}\right]\right\}}{x_0^4\left(4x_0^3+\omega_0\right)^{3/2}}. \tag{24}$$

with the dimensionless parameters defined as: $\omega_0 = M^2\omega$, and $x_0 = u_0 M$;

In the "Post-Newtonian" limit $4x_0^3/\omega_0 \ll 1$, we obtain the GR result $\delta\phi_{GR} = 6\pi b^2$, where $b^2 = M/a\,(1-e^2)$.

The observed value of the perihelion precession of the planet Mercury is $\delta\phi_{Obs} = 43.11 \pm 0.21$ arcsec per century. Therefore the range of variation of the perihelion precession is $\delta\phi_{Obs} \in (42.90, 43.32)$.

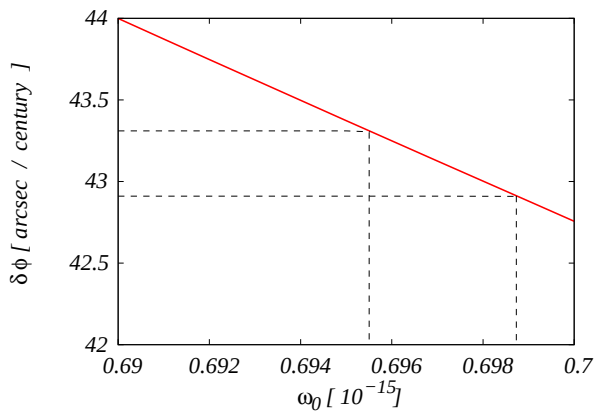

Figure 1: Variation of the precession angle $\delta\phi$ as a function of ω_0.

This range of observational values fixes the range of ω_0 as

$$\omega_0 \in \left(6.95431 \times 10^{-16}, 6.98821 \times 10^{-16}\right). \tag{25}$$

2.2 The deflection of light

The deflection angle of a light ray by the gravitational field of a massive body in a spherically symmetric geometry is given by

$$\Delta\phi = 2\left|\phi\left(r_0\right) - \phi\left(\infty\right)\right| - \pi, \tag{26}$$

and

$$\phi\left(r\right) = \phi\left(\infty\right) + \int_r^\infty \frac{e^{\lambda/2}}{\sqrt{e^{\nu(r_0)-\nu(r)}\left(r/r_0\right)^2 - 1}}\frac{dr}{r}, \tag{27}$$

where r_0 is the distance of the closest approach. Here we have taken into account that in the absence of a gravitational field a light ray propagates along a straight line, and precisely for this reason π has appeared in Eq. (26). In the case of the Sun, the deflection angle of alight ray is $\Delta\phi = 1.72752''$.

The deflection angle of light rays passing nearby the Sun in the KS's geometry:

$$\phi\left(x_0\right) = \phi\left(\infty\right) + \int_1^\infty \frac{\left[1 + \omega_0 x_0^2 x^2 - \sqrt{x_0 x(\omega_0^2 x_0^3 x^3 + 4\omega_0)}\right]^{-1/2}}{\sqrt{\left[1 + \omega_0 x_0^2 - \sqrt{x_0(\omega_0^2 x_0^3 + 4\omega_0)}\right] x^2 / \left[1 + \omega_0 x_0^2 x^2 - \sqrt{x_0 x(\omega_0^2 x_0^3 x^3 + 4\omega_0)}\right] - 1}}\frac{dx}{x}. \tag{28}$$

where $\omega = \omega_0/M^2$, $r_0 = x_0 M$, and considering $r = r_0 x$.

For the Sun, by taking $r_0 = R_\odot = 6.955 \times 10^{10}$ cm, we find for x_0 the value $x_0 = 4.71194 \times 10^5$. The variation of the deflection angle $\Delta\phi = 2\left|\phi\left(x_0\right) - \phi\left(\infty\right)\right| - \pi$ is represented, as a function of ω_0, in Fig. 2.

The best available data come from long baseline radio interferometry, which gives $\delta\phi_{LD} = \delta\phi_{LD}^{(GR)}\left(1 + \Delta_{LD}\right)$, with $\Delta_{LD} \leq 0.0017$, where $\delta\phi_{LD}^{(GR)} = 1.7275$ arcsec.

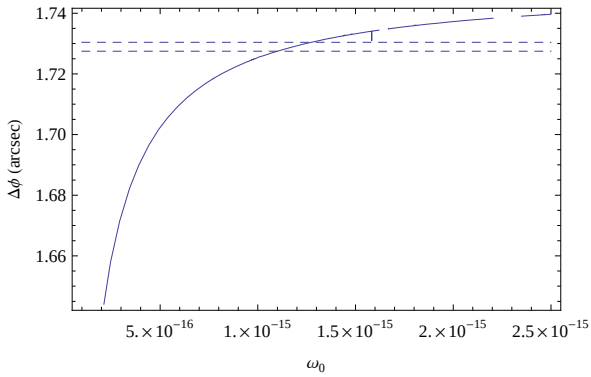

Figure 2: The light deflection angle $\Delta\phi$ (in arcseconds) as a function of the parameter ω_0.

The observational constraints of light deflection restricts the value of ω_0 to

$$\omega_0 \in \left(1.1 \times 10^{-15}, 1.3 \times 10^{-15}\right). \tag{29}$$

2.3 Radar echo delay

Measure the time required for radar signals to travel to an inner planet or satellite in two circumstances:

a) when the signal passes very near the Sun and

b) when the ray does not go near the Sun.

The time of travel of light t_0 between two planets, situated far away from the Sun, is given by $t_0 = \int_{-l_1}^{l_2} dy$, where l_1 and l_2 are the distances of the planets to the Sun, respectively.

If the light travels close to the Sun, the time travel is

$$t = \int_{-l_1}^{l_2} \frac{dy}{v} = \int_{-l_1}^{l_2} e^{[\lambda(r)-\nu(r)]/2} dy, \tag{30}$$

The time difference is

$$\Delta t = t - t_0 = \int_{-l_1}^{l_2} \left\{ e^{[\lambda(r)-\nu(r)]/2} - 1 \right\} dy = \int_{-l_1}^{l_2} \left\{ e^{\left[\lambda\left(\sqrt{y^2+R_\odot^2}\right)-\nu\left(\sqrt{y^2+R_\odot^2}\right)\right]/2} - 1 \right\} dy. \tag{31}$$

with $r = \sqrt{y^2 + R_\odot^2}$, and $R_\odot$ is the radius of the Sun.

Cassini spacecraft: For the time delay of the signals emitted on Earth, and which graze the Sun, one obtains $\Delta t_{RD} = \Delta t_{RD}^{(GR)} (1 + \Delta_{RD})$, with $\Delta_{RD} \leq (2.1 \pm 2.3) \times 10^{-5}$.

The standard GR radar echo delay value: $\Delta t_{RD}^{(GR)} \approx 4M_\odot \ln\left(4l_1 l_2 / R_\odot^2\right) \approx 2.4927 \times 10^{-4}$ s.

By introducing a new variable ξ defined as $y = 2\xi M_\odot$, and by representing again ω as $\omega = \omega_0 / M_\odot^2$, we obtain for the time delay for the KS solution, the following expression

$$\Delta t_{RD} = 16 M_\odot \omega_0 \int_{-\xi_1}^{\xi_2} \frac{(\xi^2 + a^2)\left[\sqrt{1 + (1/2\omega_0)(\xi^2 + a^2)^{-3/2}} - 1\right]}{1 - 4\omega_0 (\xi^2 + a^2)\left[\sqrt{1 + (1/2\omega_0)(\xi^2 + a^2)^{-3/2}} - 1\right]} d\xi, \tag{32}$$

where $a^2 = R_\odot^2 / 4M_\odot^2$, $\xi_1 = l_1 / 2M_\odot$, and $\xi_2 = l_2 / 2M_\odot$, respectively.

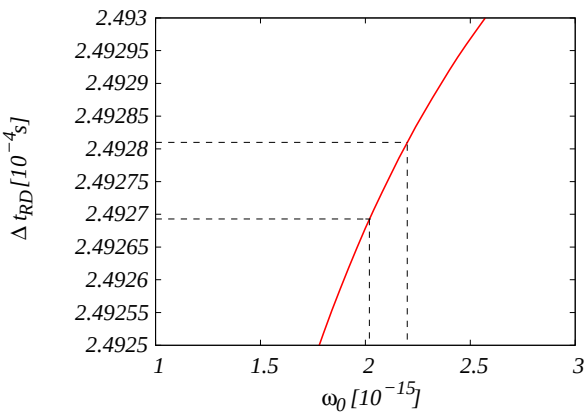

Figure 3: Variation of the time delay Δt_{RD} as a function of ω_0.

The observational values of the radar echo delay are consistent with the KS black hole solution in Hořava-Lifshitz gravity if

$$\omega_0 \in \left(2 \times 10^{-15}, 3 \times 10^{-15}\right) \tag{33}$$

3 Conclusion

In the present work, we have considered the observational and experimental possibilities for testing, at the level of the Solar System, the Kehagias and Sfetsos solution of the vacuum field equations in Hořava-Lifshitz gravity.

The parameter ω, having the physical dimensions of length^{-2}, is constrained by the perihelion precession of Mercury to a value of
$\omega = 7 \times 10^{-16}/M_{\odot}^2 = 3.212 \times 10^{-26}$ cm^{-2}.

The deflection angle of the light rays by the Sun can be fully explained in Hořava-Lifshitz gravity with the parameter ω having the value
$\omega = 10^{-15}/M_{\odot}^2 = 4.5899 \times 10^{-26}$,

while the radar echo delay experiment suggests a value of
$\omega = 2 \times 10^{-15}/M_{\odot}^2 = 9.1798 \times 10^{-26}$ cm^{-2}.
From these values we can give an estimate of ω as

$$\omega = (5.660 \pm 3.1) \times 10^{-26} \text{ cm}^{-2}. \tag{34}$$

The gravitational dynamics of the KS solution is determined by the free parameter ω. Therefore the explanation of the classical tests of GR must require a very precise fine tuning of this constant at the level of the Solar System. It is also very important for future observations to determine if ω is a local quantity or a universal constant.

By assuming that ω is a universal constant, its smallness suggests the possibility that it may have a microscopic origin. It is interesting to note that ω can be represented as a function of the fundamental physics constants $\hbar$, c, G and the mass of the electron m_e as

$$\omega \approx \frac{\hbar^7}{G^2 c^{10} m_e^9} = \frac{\lambda^7}{G^2 c^3 m_e^2} \approx 1.28 \times 10^{-26} \text{ cm}^{-2}, \tag{35}$$

This representation may indicate the possibility of an interplay between quantum and classical effects in Hořava-Lifshitz gravity, which could be effectively observed on a cosmic scale.

Thus, the study of the classical tests of general relativity provide a very powerful method for constraining the allowed parameter space of the Hořava-Lifshitz gravity solutions, and to provide a deeper insight into the physical nature and properties of the corresponding spacetime metrics.

This opens the possibility of testing Hořava-Lifshitz gravity by using astronomical and astrophysical observations at the Solar System scale. In the present work we have provided some basic theoretical tools necessary for the in depth comparison of the predictions of the Hořava-Lifshitz gravity model with the observational/experimental results.