II Workshop on Black Holes

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Zero frequency limit of gravitational radiation in high-energy collisions

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Work in progress:

E. Berti, V. Cardoso, T. Hinderer, M.L., F. Pretorius, U. Sperhake, N. Yunes

Zero Frequency Limit (ZFL) (Weinberg '64 Smarr '77)

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$h_{\mu\nu} = 4G \int \frac{S_{\mu\nu}\left(t_{ret}, \mathbf{x}'\right)}{|\mathbf{x}' - \mathbf{x}|} d^3\mathbf{x}' \qquad S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T$$

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- **Zero duration** → infinite acceleration
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- **Zero duration** → infinite acceleration
- **Hard collision** → incoming and outgoing trajectories with constant velocities

→ Valid for arbitrary velocities

→ Details of internal structure irrelevant

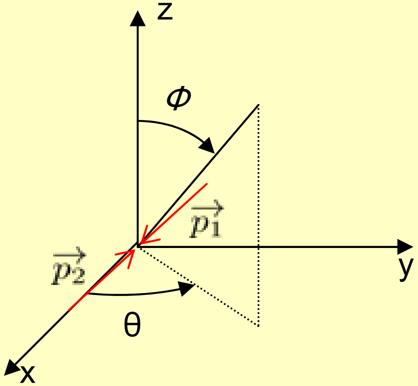
→ Flat, large memory $\frac{dE}{d\omega} \propto \left(h_{t=\infty} - h_{t=-\infty}\right)^2$

L. Smarr, Phys. Rev. D ***15,*** *2069 (1977)*

ZFL: Head-on Collision

→ Free particles, changing abruptly at t=0:

$$T^{\mu\nu}(\mathbf{x},t) = \sum_{i=1}^{2} \frac{p_i^\mu p_i^\nu}{E_i}\delta^3(\mathbf{x}-\mathbf{v}_i t)\theta(-t) + \sum_{i=1}^{2} \frac{p_i'^\mu p_i'^\nu}{E_i'}\delta^3(\mathbf{x}-\mathbf{v}_i' t)\theta(t)$$



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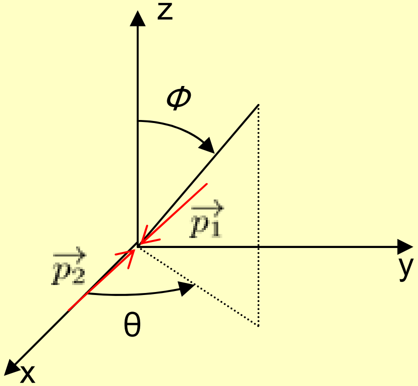

$$\frac{d^2E}{d\omega d\Omega} = 2G\omega^2 \left(T^{\mu\nu}(\mathbf{k},\omega) T^*_{\mu\nu}(\mathbf{k},\omega) - \frac{1}{2} \left| T^{\lambda}{}_{\lambda}(\mathbf{k},\omega) \right|^2 \right)$$

Center of momentum frame:

$$\frac{d^2E}{d\omega d\Omega} = \frac{G m_1^2 \gamma_1^2 v_1^2 \left(\sin^2\theta \cos^2\phi - 1\right)^2 (v_1 + v_2)^2}{4\pi^2 \left(1 - v_1 \sin\theta\cos\phi\right)^2 \left(1 + v_2 \sin\theta \cos\phi\right)^2}$$

→ Independent of ω

→ Introduce a **cutoff** frequency

L. Smarr, Phys. Rev. D ***15,*** *2069 (1977)*

ZFL: Head-on Collision

Radiated Momentum: $$\frac{dP^i}{d\omega} = \int_S \frac{d^2E}{d\omega d\Omega} n^i d\Omega$$

$$\frac{dP^i}{d\omega} = 0 \quad , i = y, z$$ → Momentum is radiated in the direction of motion

$$\begin{aligned}
\frac{dP^x}{d\omega} &= \frac{m_2^2}{\pi v_2^2(v_1+v_2)}\left(\left(v_1\left(v_2^2+3\right)\right)-v_2^3+5v_2\right)\operatorname{arctanh}(v_2)+ \\
&+ \frac{m_1^2}{\pi v_1^2(v_1+v_2)}(v_1^2(v_1-v_2)-5v_1-3v_2)\operatorname{arctanh}(v_1)+ \\
&+ \frac{\gamma_1^2 m_1^2}{\pi v_1 v_2^3}(v_1-v_2)\left(v_1^2\left(v_2^2-3\right)-5v_1v_2-3v_2^2\right)
\end{aligned}$$

ZFL: Head-on Collision

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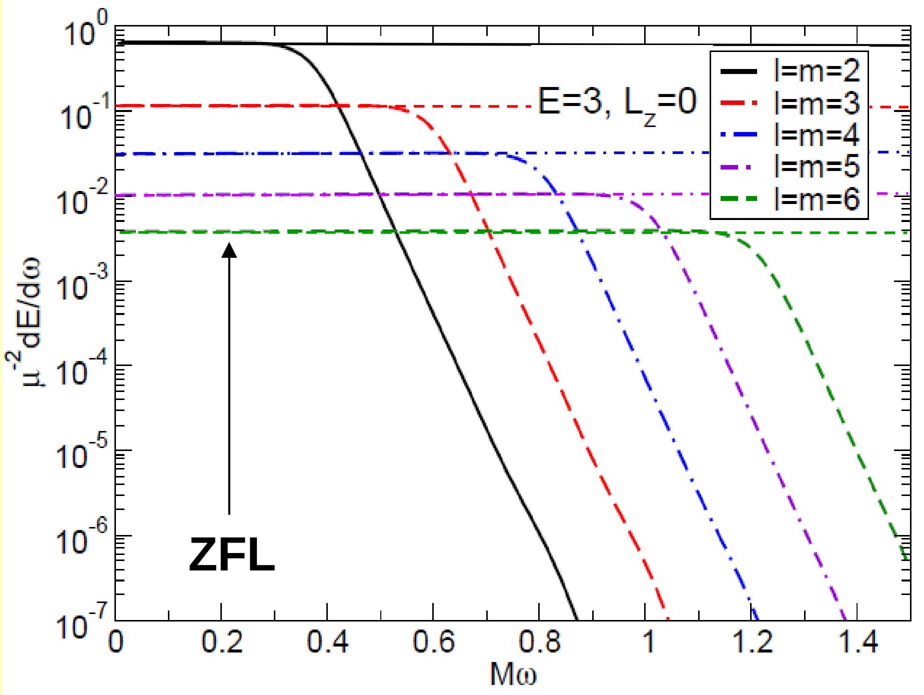
$$\begin{aligned}\frac{dP^x}{d\omega} &= \frac{m_2^2}{\pi v_2^2 (v_1 + v_2)} \left(\left(v_1 \left(v_2^2 + 3 \right) \right) - v_2^3 + 5 v_2 \right) \operatorname{arctanh}(v_2) + \\ &+ \frac{m_1^2}{\pi v_1^2 (v_1 + v_2)} (v_1^2 (v_1 - v_2) - 5 v_1 - 3 v_2) \operatorname{arctanh}(v_1) + \\ &+ \frac{\gamma_1^2 m_1^2}{\pi v_1 v_2^3} (v_1 - v_2) \left(v_1^2 \left(v_2^2 - 3 \right) - 5 v_1 v_2 - 3 v_2^2 \right) \\ &= \begin{cases} 0, & m_1 = m_1 \\ \frac{m_1^2 \gamma_1^2 \left(v_1 \left(15 - 13 v_1^2 \right) - 3 \left(v_1^4 - 6 v_1^2 + 5 \right) \operatorname{arctanh}(v_1) \right)}{3 \pi v_1^2}, & m_1 \ll m_2 \end{cases}\end{aligned}$$

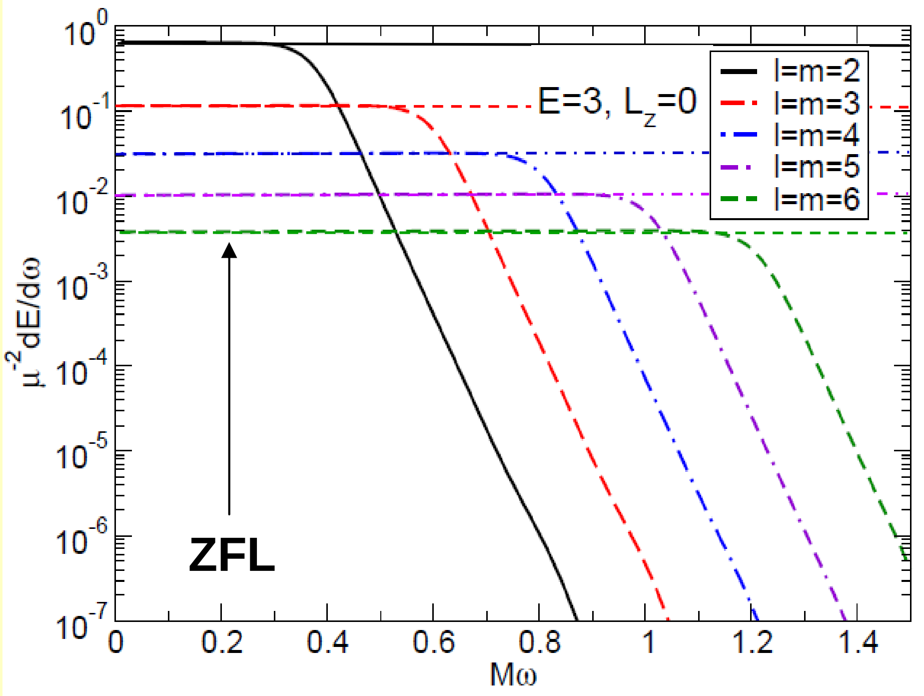
Numerical Results: Head-on Collision



Point-particle falling into a black hole

Numerical Results: Head-on Collision

U. Sperhake, V. Cardoso, F. Pretorius, E. Berti, and J.A. Gonzalez, Phys. Rev. Lett., **101**, 161101 (2008)



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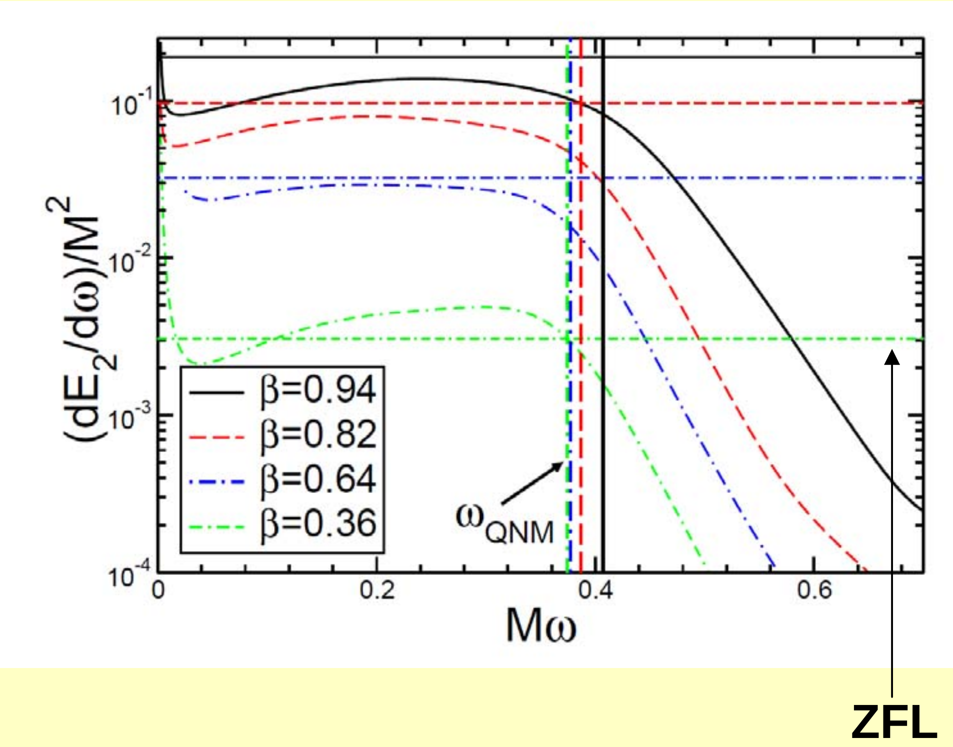
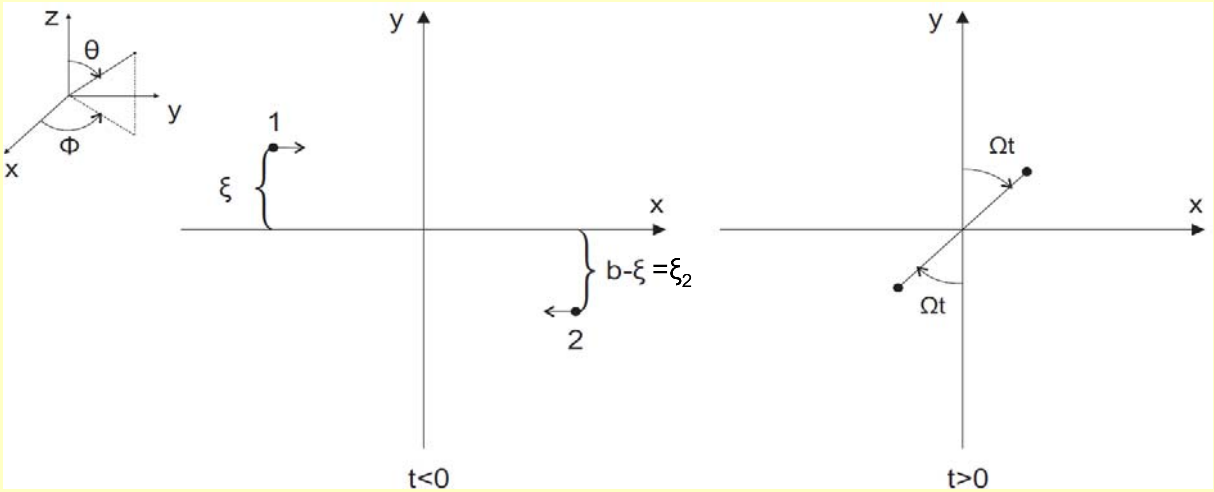


Collision of two equal mass black holes

ZFL: Non Head-on Collision

Center of momentum frame



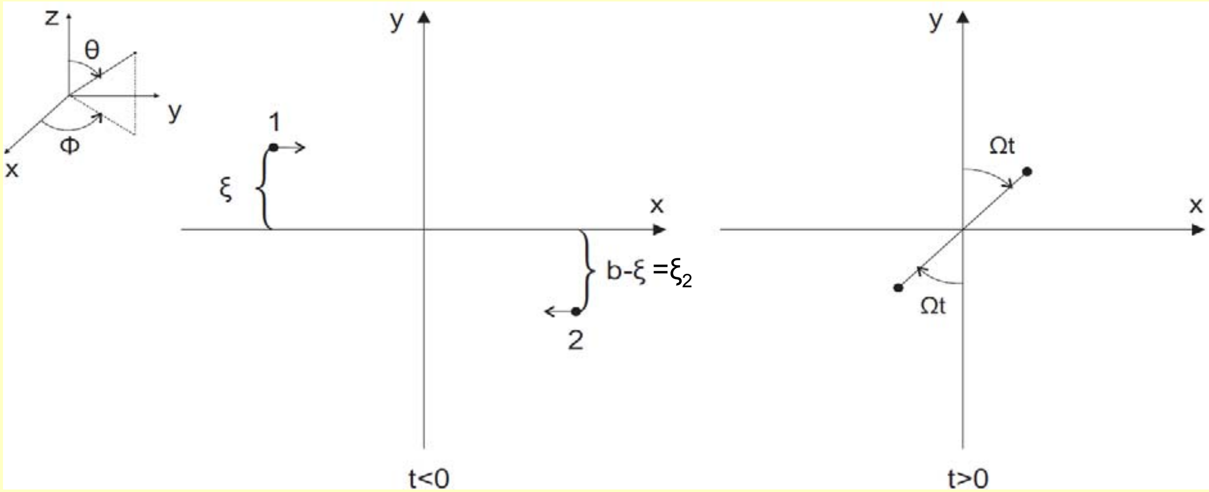
Rotation frequency:

$$\Omega = \frac{\gamma_1 m_1 v_1 \xi + \gamma_2 m_2 v_2 \xi_2}{\gamma_1 m_1 \xi^2 + \gamma_2 m_2 \xi_2^2}$$

$$\begin{aligned} T^{\mu\nu}(\mathbf{x}, t) &= \sum_{i=1}^{2} \frac{p_i^\mu p_i^\nu}{E_i} \delta^3(\mathbf{x} - \mathbf{x}_i(t))\Theta(-t) + \\ &+ \sum_{i=1}^{2} \frac{(p_i^\mu)'(t)(p_i^\nu)'(t)}{E_i'(t)} \delta^3(\mathbf{x} - \mathbf{x}_i'(t))\Theta(t) \end{aligned}$$

ZFL: Non Head-on Collision

Center of momentum frame



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$\nabla_\mu T^{\mu\nu} \neq 0 \text{ for } \nu = x, y$ → Energy momentum tensor is not conserved!

ZFL: Non Head-on Collision

Constraining forces: (Price and Sandberg '73)

$$\left\{\begin{array}{lll} T^{xx}_{\rm tens}(t,\mathbf{x}) & = & -\tau_1(r)S^2\delta(\cos\theta)\delta(\phi+\Omega t-\pi/2)\Theta(t) \\ & - & \tau_2(r)S^2\delta(\cos\theta)\delta(\phi+\Omega t-3\pi/2)\Theta(t) \\ & & \\ T^{yy}_{\rm tens}(t,\mathbf{x}) & = & -\tau_1(r)C^2\delta(\cos\theta)\delta(\phi+\Omega t-\pi/2)\Theta(t) \\ & - & \tau_2(r)C^2\delta(\cos\theta)\delta(\phi+\Omega t-3\pi/2)\Theta(t) \\ & & \\ T^{xy}_{\rm tens}(t,\mathbf{x}) & = & -\tau_1(r)S\,C\delta(\cos\theta)\delta(\phi+\Omega t-\pi/2)\Theta(t) \\ & - & \tau_2(r)S\,C\delta(\cos\theta)\delta(\phi+\Omega t-3\pi/2)\Theta(t) \end{array}\right.$$

$$\tau_1(r)=\frac{m_1\xi\Omega^2}{r^2\sqrt{1-(\xi\Omega)^2}}\,,\qquad \tau_2(r)=\frac{m_2\xi_2\Omega^2}{r^2\sqrt{1-(\xi_2\Omega)^2}}$$

*R. H. Price and V. D. Sandberg, Phys. Rev. D **8**, 1640 (1973)*

ZFL: Non Head-on Collision – equal mass collision

Radiated energy: $$\frac{d^2E}{d\omega d\Omega} = 2G\omega^2 \left(T^{\mu\nu}(\mathbf{k},\omega) T^*_{\mu\nu}(\mathbf{k},\omega) - \frac{1}{2}\left| T^{\lambda}{}_{\lambda}(\mathbf{k},\omega)\right|^2 \right)$$

Fourier transform:

$$\begin{aligned} T^{\mu\nu}(\mathbf{k},\omega) &= \frac{p_1^\mu p_1^\nu}{2\pi i E_1(\omega - v_1 k_x)} e^{-ik_y \frac{b}{2}} + \frac{p_2^\mu p_2^\nu}{2\pi i E_2(\omega + v_2 k_x)} e^{ik_y \frac{b}{2}} + \\ &+ \sum_{i=1}^{2} \int_{-\infty}^{\infty} \frac{p_i'^\mu(t) p_i'^\nu(t)}{2\pi E_i'(t)} \exp(i\omega t - i\mathbf{k}\cdot\mathbf{x}_i'(t))\Theta(t) dt + \\ &+ \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \, T^{\mu\nu}_{\text{tens}}(\mathbf{x},t) e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}} \end{aligned}$$

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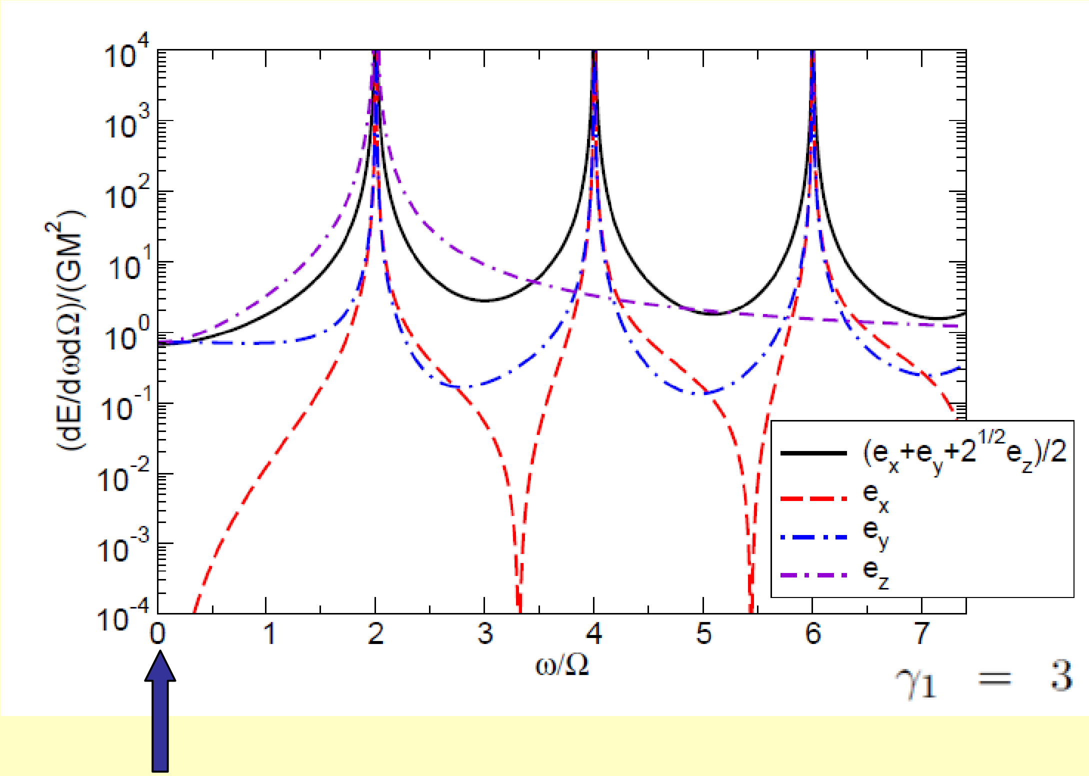
$$e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}_i'(t)} = e^{i\omega t} \underbrace{\exp\left(\pm i \frac{\omega b}{2} \sin\theta \sin(\Omega t + \phi) \right)}$$

$$e^{iz\sin(\alpha)} = \sum_{n=-\infty}^{n=+\infty} J_n(z) e^{in\alpha}$$ ⟵ Jacobi-Anger expansion

$N \geq 10$ sufficient for an accuracy of 1% or better

ZFL: Non Head-on Collision – equal mass collision



$$m_1 = m_2 \Longrightarrow \Omega = \frac{2v_1}{b}$$

- All spectra blow up for

$$\omega = 2n\Omega \ , \ n = 1, 2, \ldots$$

Except for $\hat{\mathbf{k}} = \mathbf{e}_z$

→ Only blows up at $\omega = 2\Omega$

E. Poisson, Phys. Rev. D47, 1497 (1993).

Zero frequency limit:

→ Smarr's head-on collision is recovered

$$\frac{d^2E}{d\omega d\Omega} = \frac{G m_1^2 \gamma_1^2 v_1^4 \left(\sin^2\theta\cos^2\phi - 1\right)^2}{\pi^2\left(v_1^2\sin^2\theta\cos^2\phi - 1\right)^2}$$

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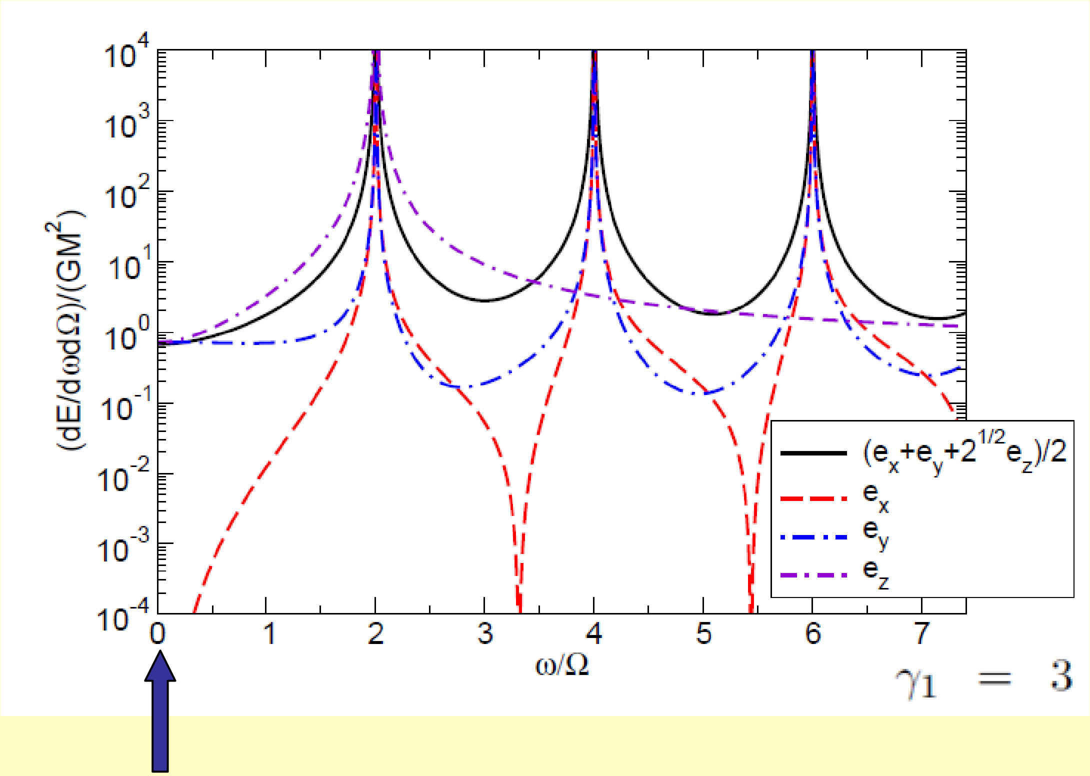

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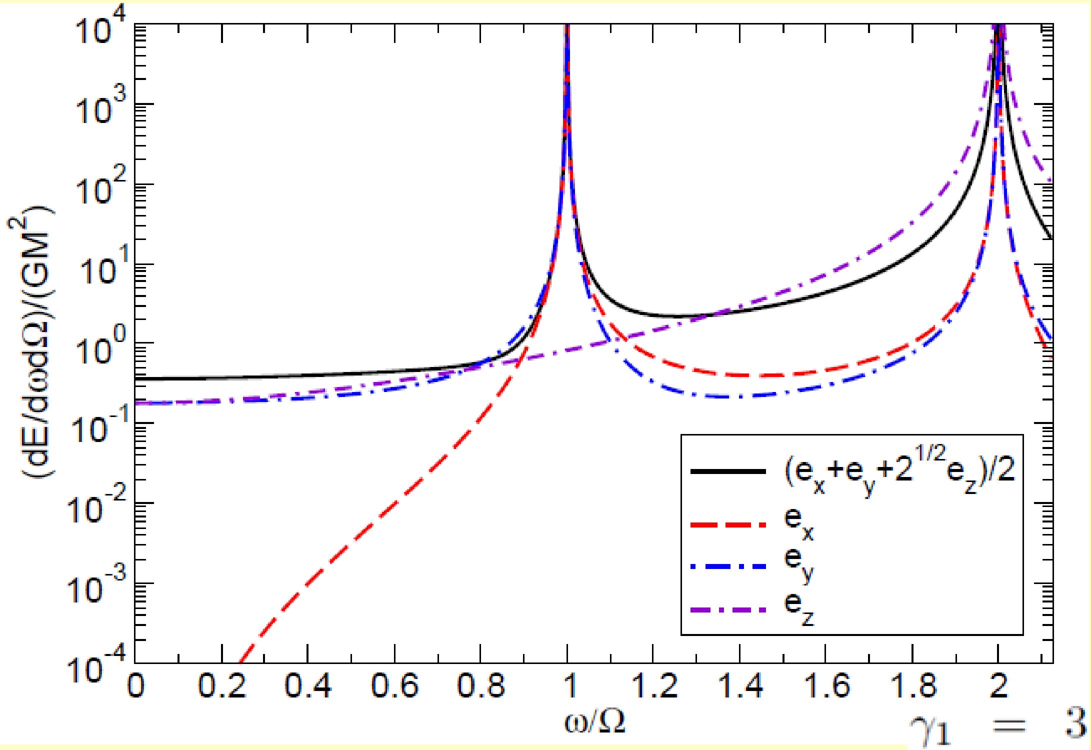
ZFL: Non Head-on Collision – extreme mass ratio



$\gamma_1 = 3$

Center of momentum frame

$$\mu \equiv m_1 \ll m_2 \equiv M$$

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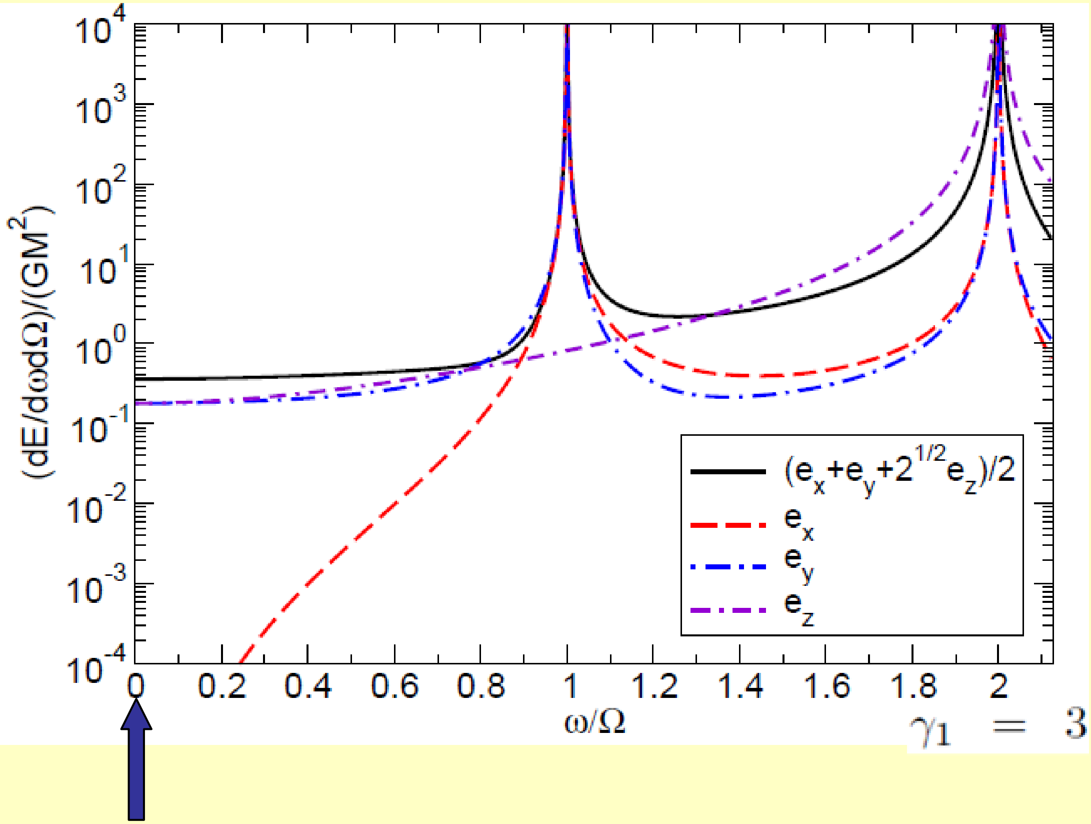
$$\Omega = \frac{\gamma_1 \mu v_1 \xi + \gamma_2 M v_2 (b - \xi)}{\gamma_1 \mu \xi + \gamma_2 M (b - \xi)}$$

$$\xi = \frac{b \gamma_2 M}{\gamma_1 \mu + \gamma_2 M}$$

ZFL: Non Head-on Collision – extreme mass ratio



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→ Smarr's head-on collision is recovered

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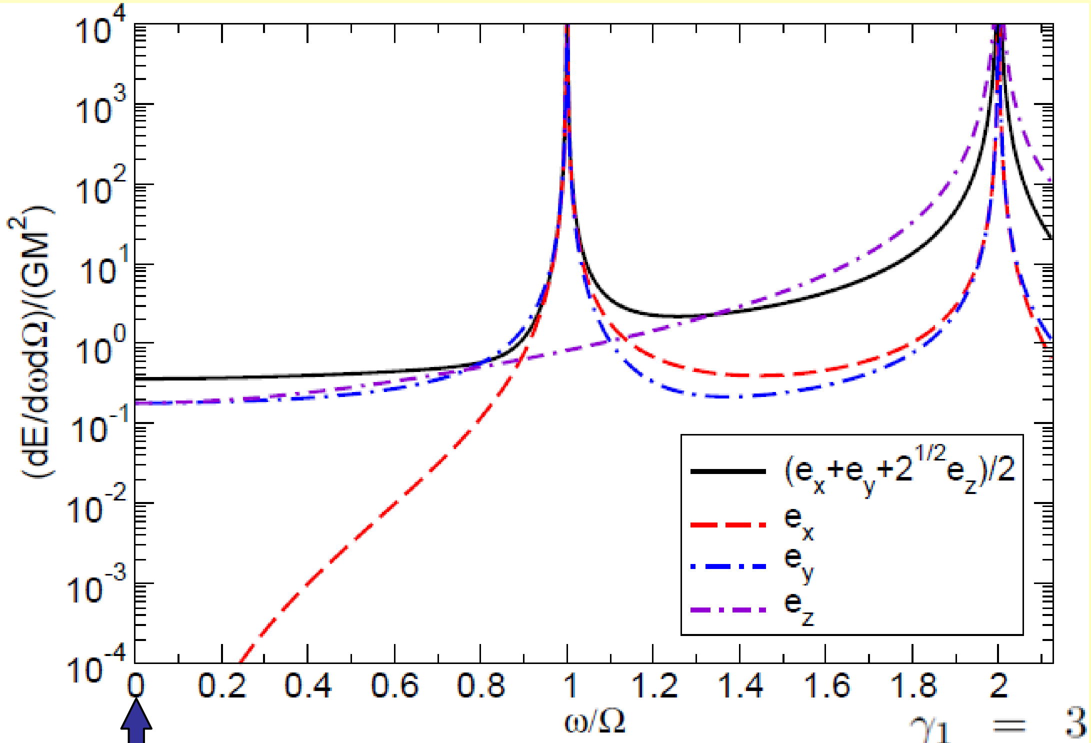
ZFL: Non Head-on Collision – extreme mass ratio



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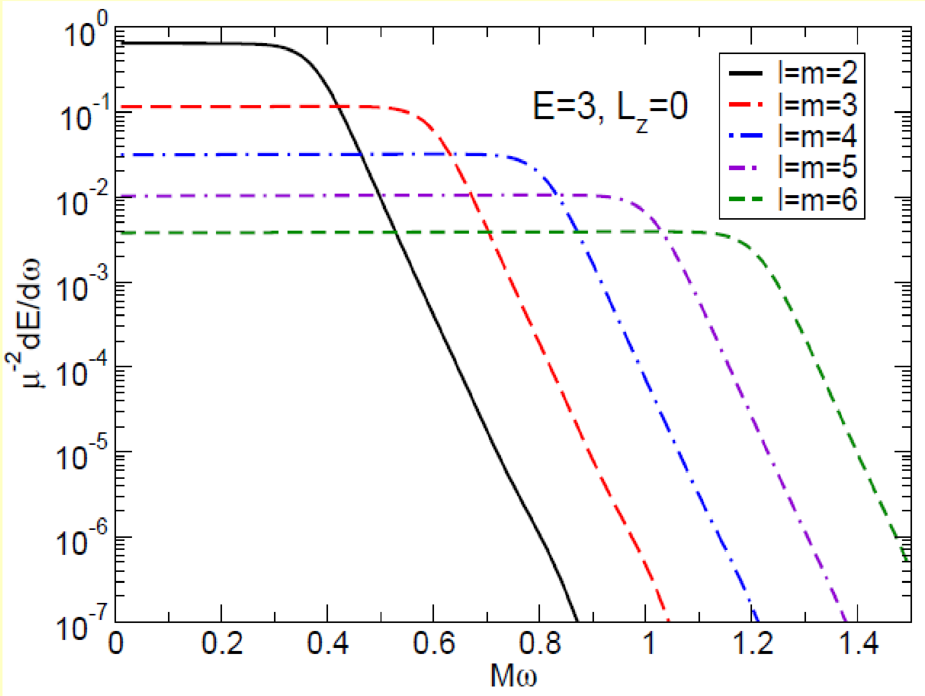
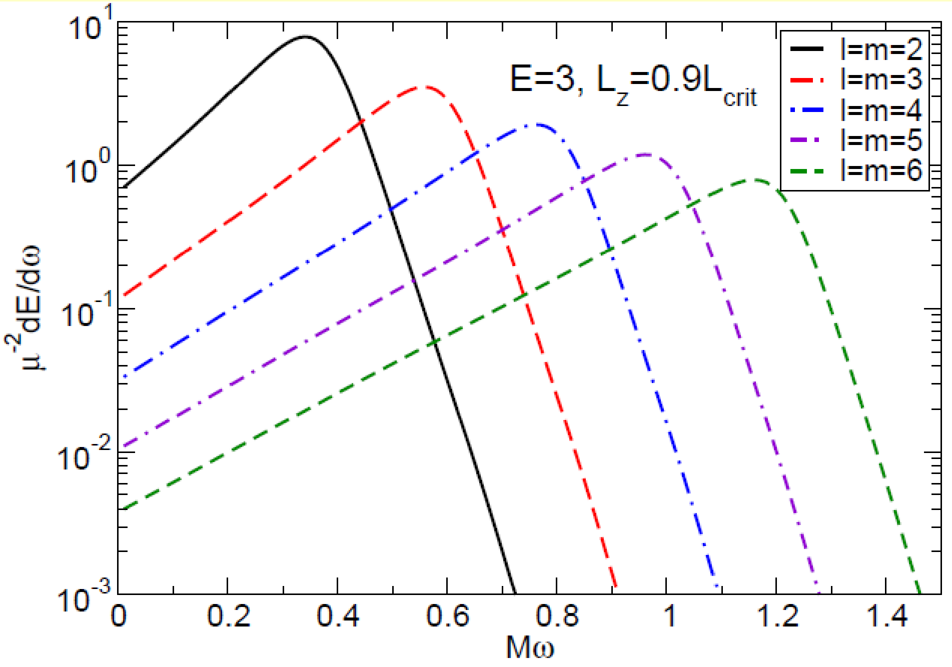
Numerical Results: Non Head-on Collision





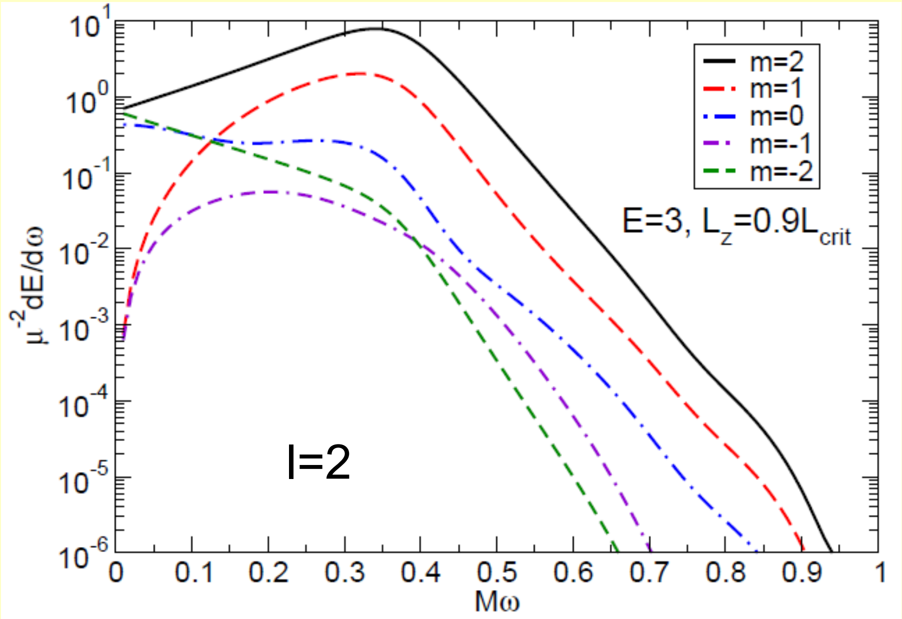
→ Head-on limit – spectrum is flat for small frequencies

→ Different behaviour near resonance
for numerical data and toy model

→ $M\omega \to 0$ limit very weakly dependent on L_Z/L_{crit}
consistent with the toy model

Numerical Results: Non Head-on Collision

	$\frac{1}{(\mu E)^2} dE_{20}^{\rm rad}/d\omega$			
$L_z/L_{\rm crit}$	$E=3$	$E=10$	$E=100$	ZFL
0.00	0.0481	0.0644	0.0663	0.0663
0.50	0.0480	0.0643	0.0662	0.0663
0.75	0.0480	0.0642	0.0662	0.0663
0.90	0.0480	0.0642	0.0661	0.0663
0.95	0.0479	0.0641	0.0661	0.0663
0.99	0.0479	0.0641	0.0661	0.0663
0.999	0.0479	0.0641	0.0661	0.0663



→ $M\omega \to 0$ limit very weakly dependent on L_Z/L_{crit}

→ Head-on case – ZFL yields $$\lim_{\omega\to 0} \frac{dE_{22}/d\omega}{dE_{20}/d\omega} = \frac{3}{2}$$

→ Good agreement with numerical results

Numerical Results: Non Head-on Collision

	$\frac{1}{(\mu E)^2} dE^{\rm rad}_{20}/d\omega$			
$L_z/L_{\rm crit}$	$E=3$	$E=10$	$E=100$	ZFL
0.00	0.0481	0.0644	0.0663	0.0663
0.50	0.0480	0.0643	0.0662	0.0663
0.75	0.0480	0.0642	0.0662	0.0663
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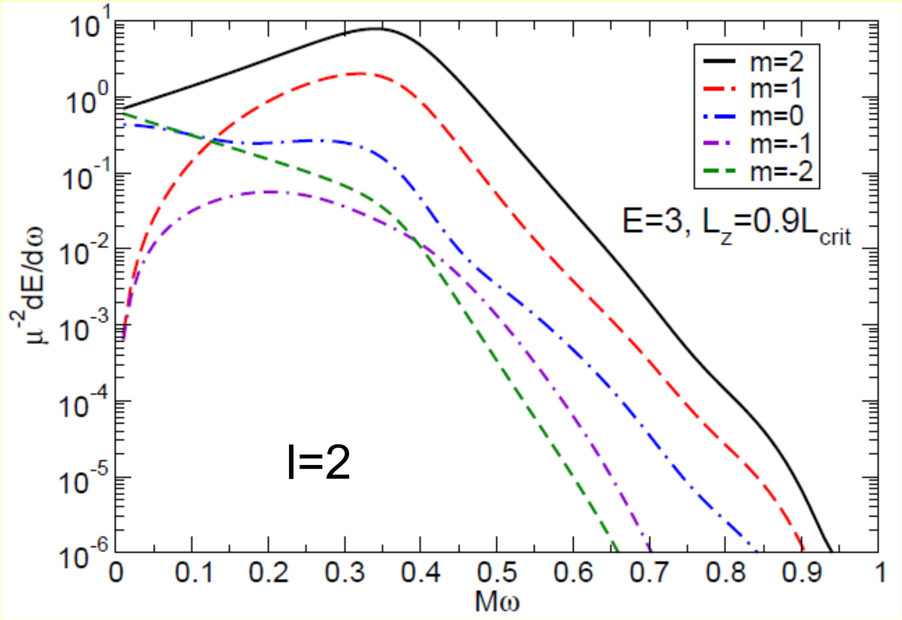


→ $m = \pm l$ have positive/negative slope for $M\omega \to 0$

→ $$\left| \frac{dE}{d\omega d\Omega} \right| \sim b$$

→ Toy model: $$\left| \frac{dE}{d\omega d\Omega} \right| \sim b^2$$

Summary

- Zero-frequency limit is independent of the impact parameter
- Numerical results weakly dependent on the impact parameter
- Different small frequency behaviour
- Not the same behaviour near resonances

Future Work

→ Dependence on the constraining forces

→ Particles with structure

→ Higher-dimensions

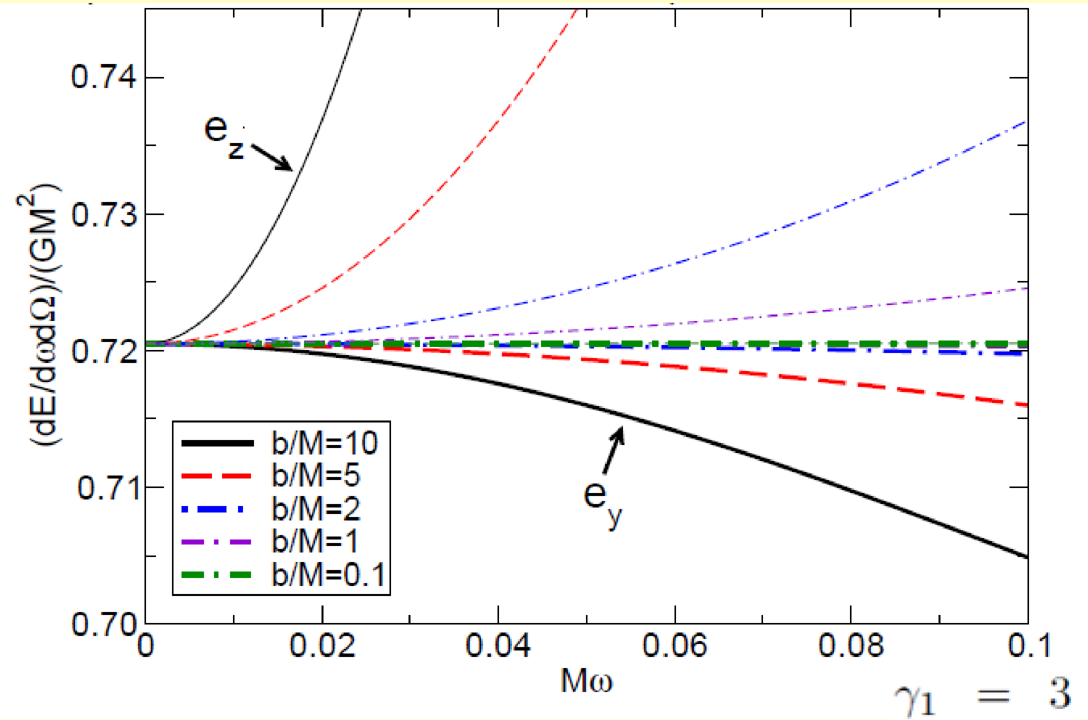
→ Radiated Momentum

→ Different ways to model the collision

Thank you

Backup Slides

ZFL: Non Head-on Collision – equal mass collision

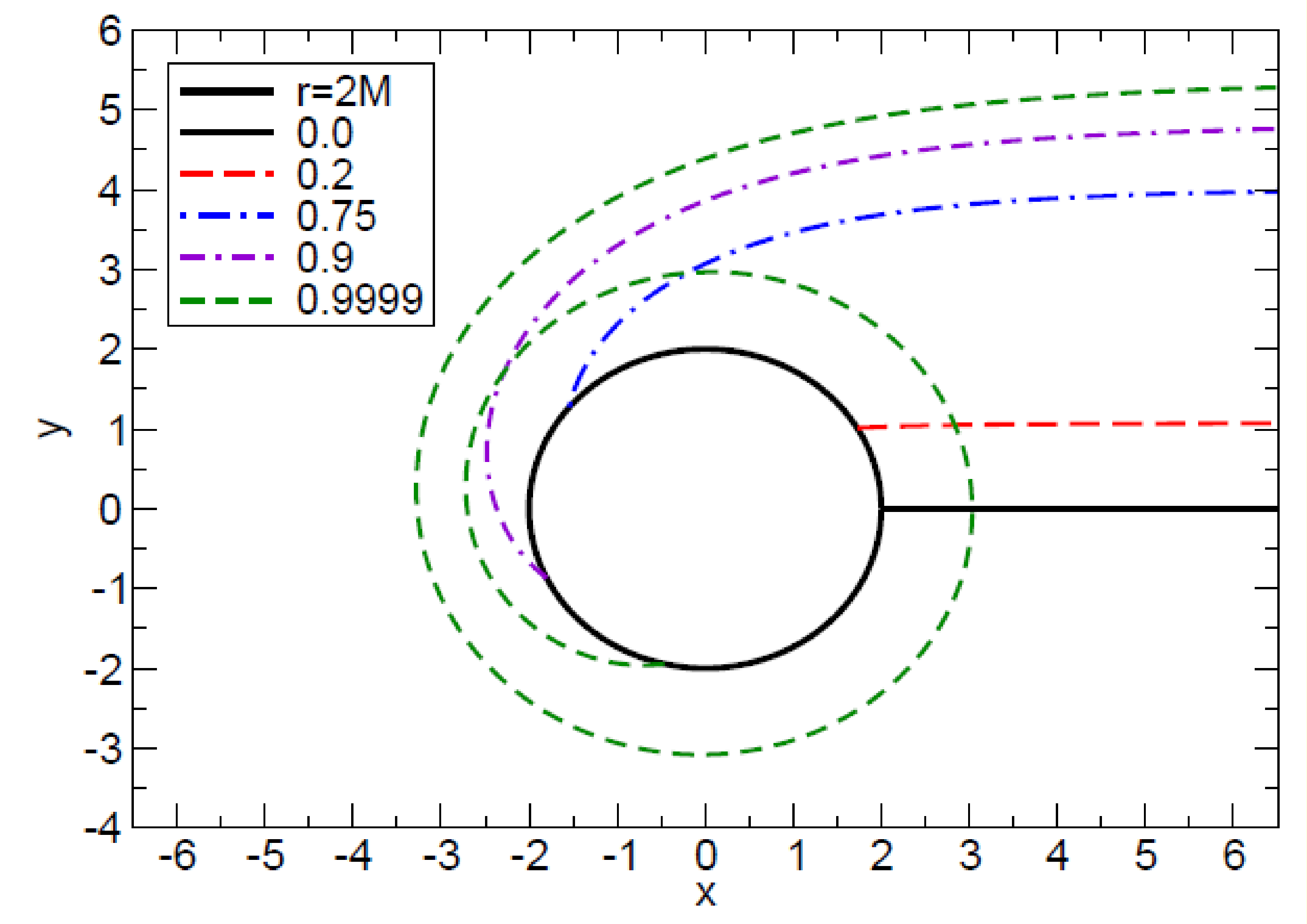


$\gamma_1 = 3$

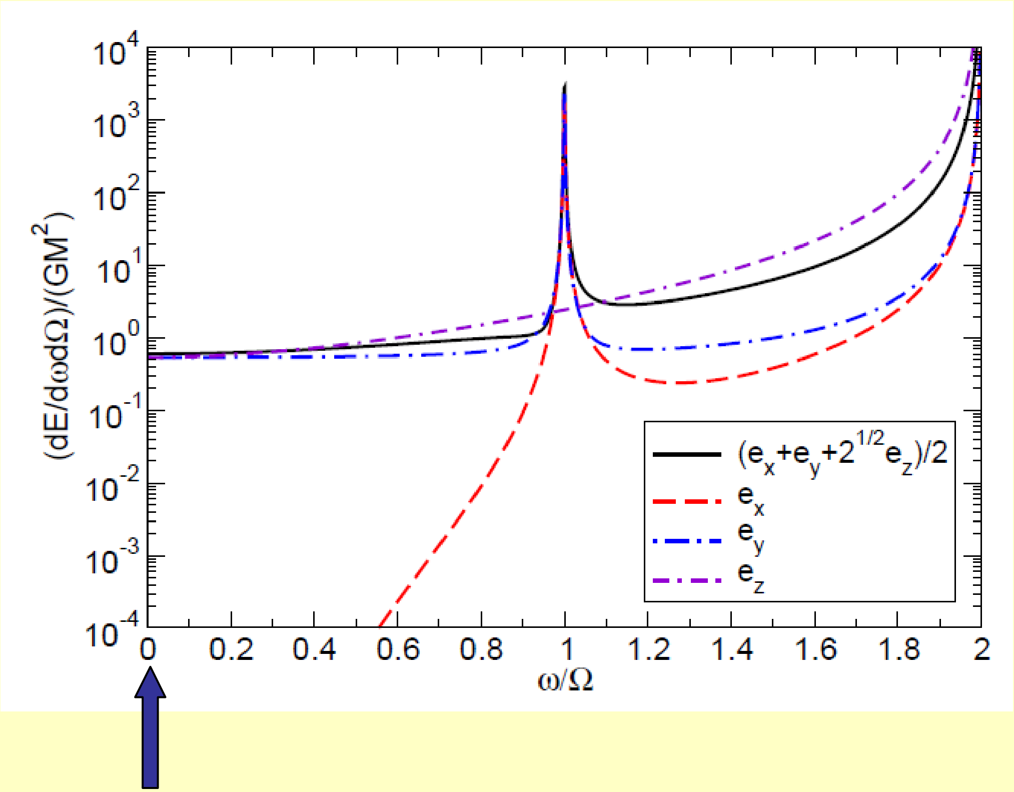
Small Mω:

$$\left.\frac{d^2E}{d\omega d\Omega}\right|_{\hat{\mathbf{k}}\neq\mathbf{e_x}} \sim (b\omega)^2 \qquad\qquad \left.\frac{d^2E}{d\omega d\Omega}\right|_{\hat{\mathbf{k}}=\mathbf{e_x}} \sim (b\omega)^4$$

Numerical Results: Non Head-on Collision



ZFL: Non Head-on Collision



- Same structure

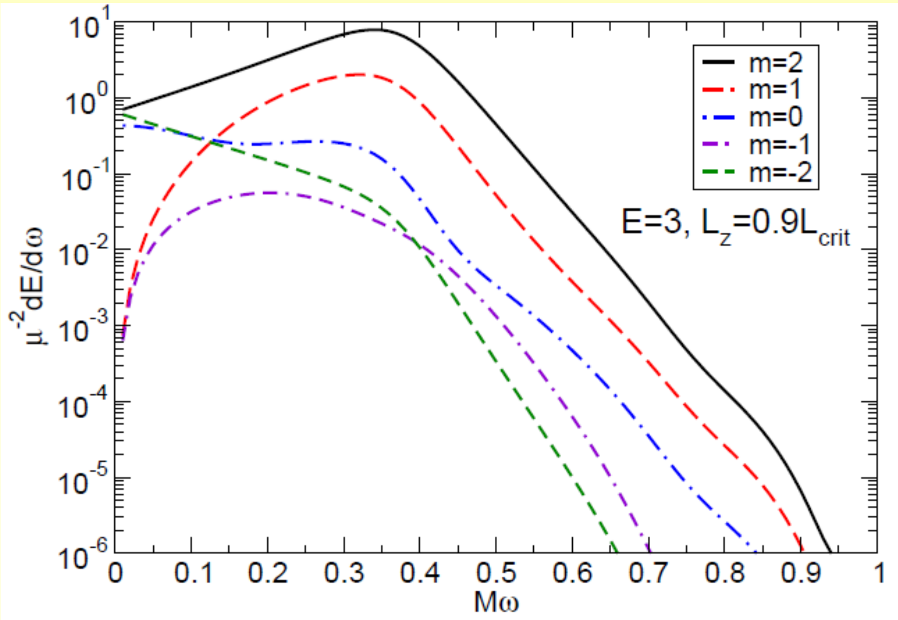
Zero frequency limit:

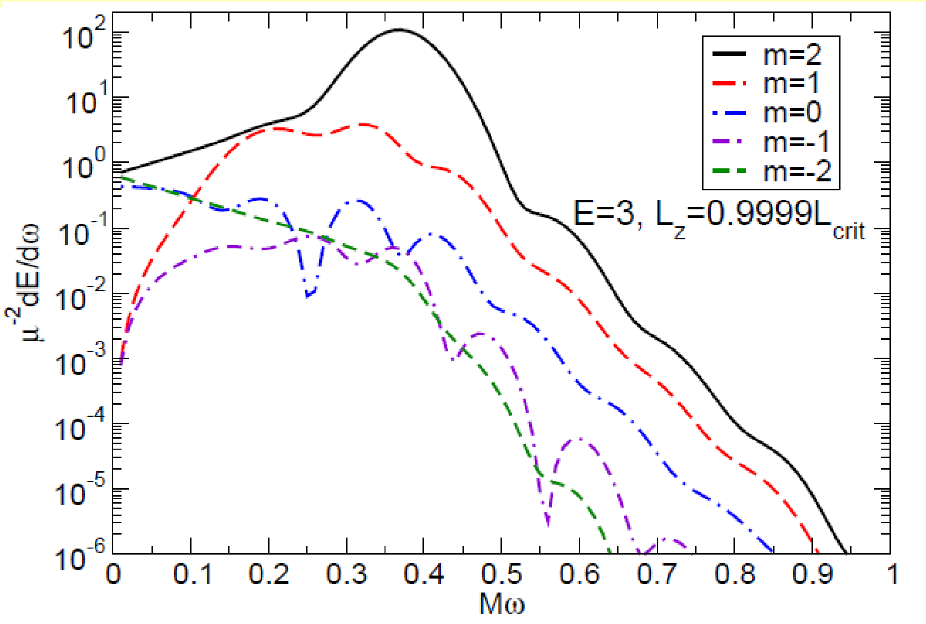
→ Smarr's head-on collision is recovered

$$\frac{d^2E}{d\omega d\Omega} = \frac{Gm^2\gamma_1^2 v_1^2 \left(\sin^2(\theta)\cos^2(\phi) - 1\right)^2 (v_1 + v_2)^2}{4\pi^2 \left(1 - v_1 \sin(\theta)\cos(\phi)\right)^2 \left(1 + v_2 \sin(\theta)\cos(\phi)\right)^2}$$

Numerical Results: Non Head-on Collision

l=2



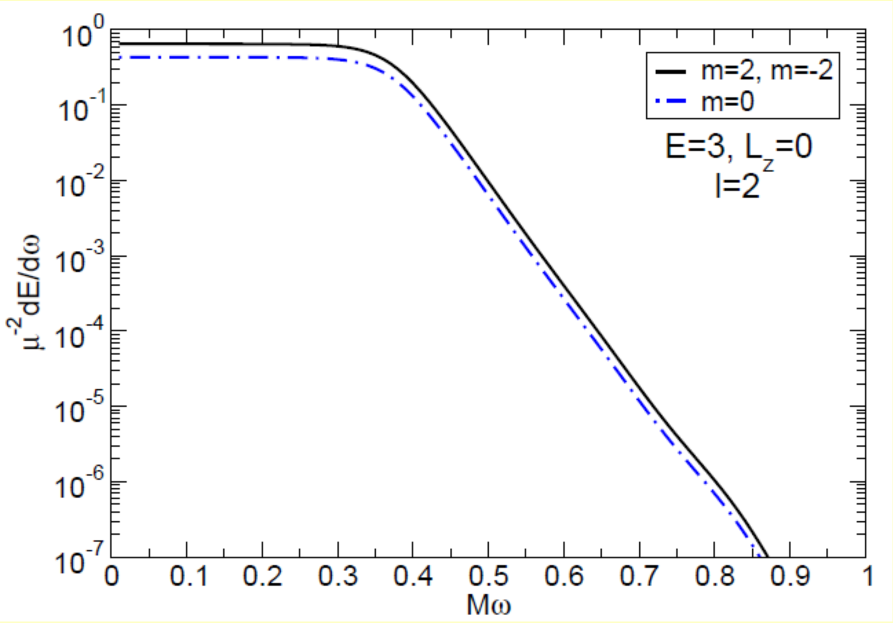


→ $m = \pm l$ have positive/negative slope for $M\omega \to 0$

→ $\left| \frac{dE}{d\omega d\Omega} \right| \sim b$

→ Toy model: $\left| \frac{dE}{d\omega d\Omega} \right| \sim b^2$

Numerical Results: Non Head-on Collision



→ Head-on case – ZFL yields $$\lim_{\omega\to 0}\frac{dE_{22}/d\omega}{dE_{20}/d\omega}=\frac{3}{2}$$

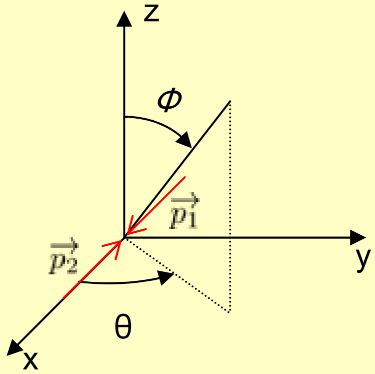
→ Good agreement with numerical results

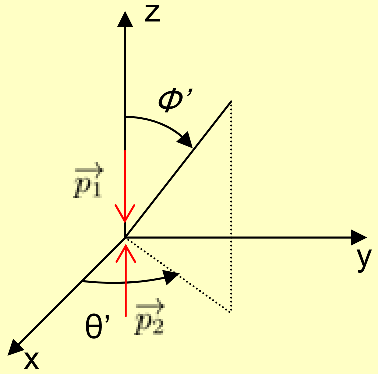
Smarr

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$\cos(\theta') = \sin(\theta)\cos(\phi)$$

$$d\Omega = d\Omega'$$





$$\frac{dE}{d\omega d\Omega} = \frac{M^2\gamma^2 v^4}{\pi^2} \frac{\sin^4\theta'}{(1 - v^2\cos^2\theta')^2}$$