On the near horizon approximation for the Schwarzschild black hole

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Abstract

An anisotropic fluid with a negative radial pressure $p = -\rho$ is supposed to exist near the horizon of a Schwarzschild black hole. The constant energy density ρ depends only on the black hole mass. The structure of the stress tensor is similar with that obtained previously for the interior region of a black hole excepting the time dependence.

Keywords : surface gravity ; anisotropic fluid ; membrane paradigm.

It is a known fact that the Schwarzschild geometry

$$ds^2 = -(1 - \frac{2m}{r})\ dt^2 + (1 - \frac{2m}{r})^{-1} dr^2 + r^2 d\Omega^2 \tag{0.1}$$

is almost flat near the event horizon $r = 2m$ [1]. For $r \approx 2m$ the line element (1) appears as

$$ds^2 = -\frac{r - 2m}{2m} dt^2 + \frac{2m}{r - 2m} dr^2 + 4m^2 d\Omega^2 \tag{0.2}$$

where m is the central mass and $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$. We use the natural units $G = c = 1$.

By using simple coordinate transformations it could be shown that (2) becomes the Rindler metric when we take $\theta, \phi = const.$ or $\Delta\theta$ and $\Delta\phi$ are negligible. We stress that only the condition $r \approx 2m$ is not enough to obtain Rindler's spacetime which is flat and, in addition, it has no spherical symmetry as Schwarzschild 's. We actually may travel around the black hole near the horizon and feel its curvature. One may check that the line element (2) has one nonzero component of the Riemann tensor : $R^{\theta\phi}{}_{\theta\phi} = 1/4m^2$.

Another way to get Rindler's geometry is to let Schwarzschild's mass m to go to infinity [2]. In this case the gravitational radius $2m$ is much larger than the domain where the measurement is performed and the horizon appears to be flat. This is even a necessary condition for to get a flat geometry near the horizon. We

have indeed that the scalar curvature $R = 2/a^2$ (a - the horizon radius) tends to zero only when m goes to infinity. Moreover, for the Schwarzschild metric the Kretschmann scalar $R^{\alpha\beta\mu\nu}{}_{\alpha\beta\mu\nu} = 12a^2/r^6$ grows when we approach the horizon from infinity (where the geometry is Minkowskian). That is also valid for the components of the Riemann tensor. But near $r = a$ the Kretschmann scalar is proportional to a^{-4} and we must take $a \to \infty$ to get a flat space.

The spacetime (2) is not, of course, an exact solution of the vacuum Einstein equations. We are therefore interested to find what source we need on the r.h.s. of Einstein's equations in order the line element (2) to be an exact solution. A similar paradigm was proposed by Figueras et al. [3] in their study of the event and apparent horizons for the Bjorken flow geometry. The nonzero Christoffel symbols for (2) are given by

$$\Gamma^r_{tt} = \frac{a-r}{2a^2}, \quad \Gamma^t_{rt} = -\frac{1}{2(a-r)}, \quad \Gamma^r_{rr} = \frac{1}{2(a-r)}$$
$$\Gamma^\phi_{\theta\phi} = cot\theta, \quad \Gamma^\theta_{\phi\phi} = -sin\theta cos\theta. \tag{0.3}$$

The Ricci tensor has only two non-vanishing components

$$R^\theta_\theta = R^\phi_\phi = \frac{1}{a^2} \tag{0.4}$$

The Einstein equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ lead to the following nonzero components of the stress tensor

$$T^t_t = T^r_r = -\frac{1}{8\pi a^2}. \tag{0.5}$$

We notice that the components (5) resembles those obtained in [4] for a time dependent anisotropic fluid inside a black hole. They emerge therefore from

$$T_{\alpha\beta} = \rho u_\alpha u_\beta + p s_\alpha s_\beta, \tag{0.6}$$

with $\rho = 1/8\pi a^2$ the rest energy density , $p = -\rho$ is the radial pressure and

$$u_\alpha = (-\sqrt{\frac{r-a}{a}}, 0, 0, 0), \qquad s_\alpha = (0, \sqrt{\frac{a}{r-a}}, 0, 0) \tag{0.7}$$

are the fluid four velocity and the unit spacelike vector in the direction of the anisotropy, respectively. We have $u_\alpha u^\alpha = -1,\ s_\alpha s^\alpha = 1\ and\ u_\alpha s^\alpha = 0$.

Let us observe that (6) is a particular case of the more general stress tensor for a fluid with nonzero transversal pressures [5] [6]

$$\tau_{\alpha\beta} = (\rho + p_\perp) u_\alpha u_\beta + p_\perp g_{\alpha\beta} + (p - p_\perp) s_\alpha s_\beta \tag{0.8}$$

We want to see now where the energy originating from (5) is concentrated. Since our geometry (2) is valid near the Schwarzschild horizon, it seems reasonable to locate it on the horizon. We remind here the so -called "membrane paradigm" [7] [8] [9] where the black hole horizon is viewed as a stretched membrane with negative bulk viscosity and a nonzero shear viscosity. Moreover, in our case the

equation of state is similar with the one obtained for the anisotropic fluid inside a black hole [4], excepting the time dependence.

The kinematical parameters associated to the velocity field u_α from (7) could be calculated using the corresponding covariant expressions for the scalar expansion, shear and vorticity tensors and the acceleration of the fluid worldlines [10]. We found that only the acceleration is nonzero, with $a^r = 1/2a = 1/4m$, which agrees with the expression for the surface gravity on the horizon of our metric (2). The shear and bulk viscosity coefficients vanish since u_α corresponds to a static observer and we know that a state of equilibrium can exist only in absence of any viscous stresses [11].

Assuming that the gravitational energy is located on a thin shell (membrane) at $r = a$, the surface energy density σ could be computed from the Gauss law

$$g = 4\pi\sigma, \tag{0.9}$$

where g is the intensity of the gravitational field (the surface gravity in our situation). With $g = 1/4m$, Eq. (9) yields $\sigma = 1/16\pi m$, an expression which could have been obtained from the Eq.(5) for the energy density $\rho = 1/32\pi m^2$, using its form in terms of the Dirac δ - function. Hence, the energy is given by $W = \sigma 4\pi a^2 = m$, as it should be.

Another way to check the previous result is by means of the Padmanabhan formula [12]

$$4\pi M = \int N s_\alpha a^\alpha \sqrt{\Sigma}\, d\theta d\phi \tag{0.10}$$

where $N = \sqrt{(r-a)/a}$ is the lapse function, Σ is the determinant of the metric on the two-surface $t = const.$, $r \approx a$ (taking the limit, Σ becomes the black hole horizon, namely the boundary of the spacetime). By using the expressions for a^α and s_α from (7), we find that M = m.

To summarize, we have shown that the Rindler metric is obtained near the horizon only when $\Delta\theta$ and $\Delta\phi$ are negligible or when the Schwarzschild mass tends to infinity. Preserving the spherical symmetry, the spacetime (2) is not an exact solution of the vacuum Einstein equations. Therefore, a source is needed on the r.h.s. of them, corresponding to an anisotropic fluid with the equation of state $p + \rho = 0, p\ and\ \rho$ depending only on the black hole mass.

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