# Motion of gyroscopes around Schwarzschild and Kerr BH – exact gravito-electromagnetic analogies

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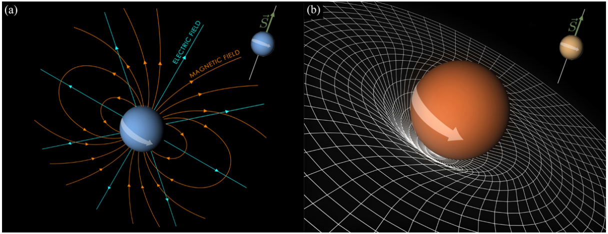

# Introduction

- It is known, since the works of Mathisson and Papapetrou that spinning particles follow worldlines which are not geodesics;
- In linearized theory, the gravitational force exerted on a spinning pole-dipole test particle (hereafter a gyroscope) takes a form: $\vec{F}_G = \nabla(\vec{B}_G.\vec{S})$ similar to the electromagnetic force on a magnetic dipole (Wald 1972).
  - But only if the gyroscope is at "rest" in a *stationary*, *weak* field!
- This analogy may be cast in an exact form (Natário, 2007) using the "Quasi-Maxwell" formalism, which holds if the gyroscope's 4-velocity is a Killing vector of a *stationary* spacetime.

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  - But only if the gyroscope is at "rest" in a *stationary*, *weak* field!
- This analogy may be cast in an exact form (Natário, 2007) using the "Quasi-Maxwell" formalism, which holds if the gyroscope's 4-velocity is a Killing vector of a *stationary* spacetime.
- There is an exact, covariant and fully general analogy relating the two forces, which is made explicit in the tidal tensor formalism (Costa & Herdeiro 2008).
- We will exemplify how this analogy provides new intuition for the understanding of spin curvature coupling.

# Force on Magnetic Dipole

$$F^{\alpha}_{EM} = \frac{DP^{\alpha}}{d\tau} = \frac{1}{2} F_{\mu\nu}{}^{;\alpha} Q^{\mu\nu}$$

- $F_{\mu\nu} \equiv$ Maxwell tensor
- $Q_{\mu\nu} \equiv$ dipole moment tensor

$$Q^{\mu\nu} = \begin{bmatrix} 0 & -d^x & -d^y & -d^z \\ d^x & 0 & \mu^z & -\mu^y \\ d^y & -\mu^z & 0 & \mu^x \\ d^z & \mu^y & -\mu^x & 0 \end{bmatrix}$$

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- In Relativity, electric and magnetic dipole moments do not exist as independent entities;
- $\vec{d}$ and $\vec{\mu}$ are the time and space components of the dipole moment 2-Form.

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- For a magnetic dipole ($\vec{d} = 0$ in its proper frame):

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- $S^{\mu\nu} \equiv$ Spin tensor;
- $\sigma \equiv$ gyromagnetic ratio ($= q/2m$ for classical spin)
- $$F^{\alpha}_{EM} = \frac{DP^{\alpha}}{d\tau} = \frac{1}{2} \sigma F_{\mu\nu}{}^{;\alpha} S^{\mu\nu}$$

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$$F^{\alpha}_{EM} = \frac{DP^{\alpha}}{d\tau} = \frac{1}{2}\sigma F_{\mu\nu}{}^{;\alpha} S^{\mu\nu}$$

- If Pirani supplementary condition $S^{\mu\nu}U_{\nu} = 0$ holds, then $S^{\mu\nu} = \epsilon^{\mu\nu\tau\lambda} S_{\tau} U_{\lambda}$
- $S^{\alpha} \equiv$ spin 4-vector; defined as the vector that, in the particle's proper frame, $S^{\alpha} = (0, \vec{S})$

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$$F^{\alpha}_{EM} = \frac{DP^{\alpha}}{d\tau} = \sigma\epsilon_{\mu\nu}{}^{\tau\lambda}F^{\mu\nu;\alpha}U_{\lambda}S_{\tau} = \sigma B_{\beta}{}^{\alpha}S^{\beta}$$

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- $B_{\alpha\beta} \equiv \star F_{\alpha\gamma;\beta} U^\gamma \equiv$ magnetic tidal tensor
  - Measures the tidal effects produced by the magnetic field $B^\alpha = \star F^\alpha_{\ \gamma} U^\gamma$ seen by the dipole of 4-velocity $U^\gamma$.

# Force on Magnetic Dipole

$$F^{\alpha}_{EM} = \sigma B_{\beta}{}^{\alpha} S^{\beta}$$

- Covariant generalization of the usual 3-D expression (valid *only* in a frame where the dipole is *at rest*!):

$$\vec{F}_{EM} = \nabla(\vec{\mu}.\vec{B})$$

- Yields the force exerted on a magnetic dipole moving with arbitrary velocity.

# Force on Gyroscope

Papapetrou equation:

$$F_G^\alpha \equiv \frac{DP^\alpha}{D\tau} = -\frac{1}{2} R^\alpha{}_{\beta\mu\nu} U^\beta S^{\mu\nu}$$

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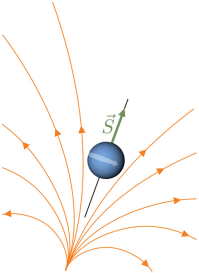
$$\frac{DP^\alpha}{D\tau} = \frac{1}{2} \epsilon_{\mu\nu}{}^{\tau\lambda} R^{\mu\nu\alpha\beta} U_\lambda U_\beta S_\tau = -\mathbb{H}_\beta{}^\alpha S^\beta$$

- $\mathbb{H}_{\alpha\beta} \equiv$"Magnetic part of the Riemann tensor"

# Magnetic-type Tidal Tensors

The electromagnetic force exerted on a magnetic dipole and the gravitational force causing the non-geodesic motion of a spinning test particle are analogous tidal effects:




- Electromagnetic Force on a Magnetic Dipole (Covariant form for $\vec{F}_{EM} = \nabla(\vec{\mu}.\vec{B})$)

$$F^{\alpha}_{EM} \equiv \frac{DP^{\alpha}}{D\tau} = \sigma B_{\gamma}^{\ \alpha} S^{\gamma} \, , \qquad B^{\alpha}_{\ \gamma} \equiv \star F^{\alpha}_{\ \beta;\gamma} U^{\beta}$$

- Gravitational Force on a Gyroscope (Papapetrou-Pirani equation)

$$F^{\alpha}_{G} \equiv \frac{DP^{\alpha}}{D\tau} = -\mathbb{H}_{\gamma}^{\ \alpha} S^{\gamma} \, , \quad \mathbb{H}^{\alpha}_{\ \gamma} \equiv \star R^{\alpha}_{\ \beta\gamma\sigma} U^{\beta} U^{\sigma}$$

- Suggests the physical analogy: $B_{\alpha\beta} \longleftrightarrow \mathbb{H}_{\alpha\beta}$
  - $B_{\alpha\beta} \equiv$ *magnetic tidal tensor;*
  - $\mathbb{H}_{\alpha\beta} \equiv$ *gravito-magnetic tidal tensor.*

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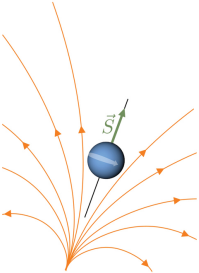


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- $\sigma = \mu/S \equiv$ gyromagnetic ratio $\Rightarrow$ equals 1 for gravity $\Rightarrow \vec{\mu} \leftrightarrow \vec{S}$

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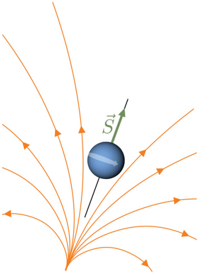


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- Relative minus sign: mass/charges of the same sign attract/repel one another ⇒ antiparallel charge/mass currents repel/attract.

| Magnetic Tidal Tensor | Gravito-Magnetic Tidal tensor |
|---|---|
| Antisymmetric part: $B_{[\alpha\beta]} = \frac{1}{2} \star F_{\alpha\beta;\gamma} U^{\gamma} - 2\pi \epsilon_{\alpha\beta\sigma\gamma} j^{\sigma} U^{\gamma}$ <br> ▶ Covariant form for $\nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + 4\pi \vec{j}$ | Antisymmetric part: $\mathbb{H}_{[\alpha\beta]} = -4\pi \epsilon_{\alpha\beta\sigma\gamma} J^{\sigma} U^{\gamma}$ |
| Trace: $B^{\alpha}{}_{\alpha} = 0$ <br> ▶ Covariant form for $\nabla \cdot \vec{B} = 0$ | Trace: $\mathbb{H}^{\alpha}{}_{\alpha} = 0$ |

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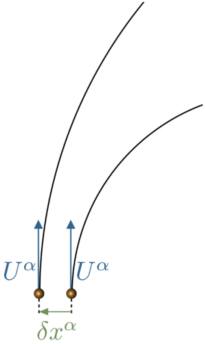
| $B_{[\alpha\beta]} = \frac{1}{2} \star F_{\alpha\beta;\gamma} U^{\gamma} - 2\pi\epsilon_{\alpha\beta\sigma\gamma} j^{\sigma} U^{\gamma}$ | $\mathbb{H}_{[\alpha\beta]} = -4\pi\epsilon_{\alpha\beta\sigma\gamma} J^{\sigma} U^{\gamma}$ |
| ▶ Space projection of Maxwell equations: | ▶ Time-Space projection of Einstein equations: |
| $F^{\alpha\beta}{}_{;\beta} = J^{\beta}$ | $R_{\mu\nu} = 8\pi \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\alpha}{}_{\alpha} \right)$ |
| Trace: | Trace: |
| $B^{\alpha}{}_{\alpha} = 0$ | $\mathbb{H}^{\alpha}{}_{\alpha} = 0$ |
| ▶ Time projection of Bianchi Identity: | ▶Time-Time projection of Bianchi Identity: |
| $\star F^{\alpha\beta}{}_{;\beta} = 0$ | $\star R^{\gamma\alpha}{}_{\gamma\beta} = 0$ |

# Electric-type Tidal Tensors

Electric-type tidal forces are described in an invariant way through the wordline deviation equations:




- Electromagnetic

$$\frac{D^2\delta x^\alpha}{D\tau^2} = \frac{q}{m} E^\alpha{}_\gamma \delta x^\gamma \, , \qquad E^\alpha{}_\gamma \equiv F^\alpha{}_{\beta;\gamma} U^\beta$$

- Gravitational (geodesic deviation)

$$\frac{D^2\delta x^\alpha}{D\tau^2} = -\mathbb{E}^\alpha{}_\gamma \delta x^\gamma \, , \qquad \mathbb{E}^\alpha{}_\gamma \equiv R^\alpha{}_{\beta\gamma\sigma} U^\beta U^\sigma$$

which yield the acceleration of the vector $\delta x^\alpha$ connecting two particles with the *same* (Ciufolini, 1986) 4-velocity $U^\alpha$ — and the same $q/m$ ratio in the electromagnetic case.
(Notation: $F_{\alpha\beta} \equiv$ Maxwell tensor, $R_{\alpha\beta\gamma\sigma} \equiv$ Riemann tensor)

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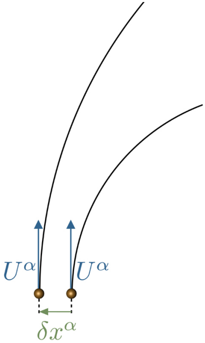


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- Suggests the physical analogy: $E_{\alpha\beta} \longleftrightarrow \mathbb{E}_{\alpha\beta}$
- $E_{\alpha\beta}$ is the covariant derivative of the electric field $E^\alpha = F^{\alpha\mu} U_\mu$ measured by the observer with (fixed) 4-velocity $U^\alpha$;

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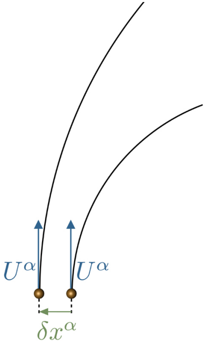


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- Suggests the physical analogy: $E_{\alpha\beta} \longleftrightarrow \mathbb{E}_{\alpha\beta}$
- Hence:
    - $E_{\alpha\beta} \equiv$ *electric tidal tensor;* $\mathbb{E}_{\alpha\beta} \equiv$ *gravito-electric tidal tensor.*

## Analogy based on tidal tensors (Costa-Herdeiro 2008)

| Electromagnetism | Gravity |
|---|---|
| Worldline deviation: $\frac{D^2\delta x^\alpha}{D\tau^2} = \frac{q}{m} E^\alpha{}_\beta \delta x^\beta$ | Geodesic deviation: $\frac{D^2\delta x^\alpha}{D\tau^2} = -\mathbb{E}^\alpha{}_\beta \delta x^\beta$ |
| Force on magnetic dipole: $\frac{DP^\beta}{D\tau} = \frac{q}{2m} B^{\alpha\beta} S_\alpha$ | Force on gyroscope: $\frac{DP^\beta}{D\tau} = -\mathbb{H}^{\alpha\beta} S_\alpha$ |
| Maxwell Equations: | Eqs. Grav. Tidal Tensors: |
| $E^\alpha{}_\alpha = 4\pi\rho_c$ | $\mathbb{E}^\alpha{}_\alpha = 4\pi\left(2\rho_m + T^\alpha{}_\alpha\right)$ |
| $E_{[\alpha\beta]} = \frac{1}{2} F_{\alpha\beta;\gamma} U^\gamma$ | $\mathbb{E}_{[\alpha\beta]} = 0$ |
| $B^\alpha{}_\alpha = 0$ | $\mathbb{H}^\alpha{}_\alpha = 0$ |
| $B_{[\alpha\beta]} = \frac{1}{2} \star F_{\alpha\beta;\gamma} U^\gamma - 2\pi\epsilon_{\alpha\beta\sigma\gamma} j^\sigma U^\gamma$ | $\mathbb{H}_{[\alpha\beta]} = -4\pi\epsilon_{\alpha\beta\sigma\gamma} J^\sigma U^\gamma$ |

# The Gravitational analogue of Maxwell's Equations

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| $E_{[\alpha\beta]} = \frac{1}{2}F_{\alpha\beta;\gamma}U^{\gamma}$ | $\mathbb{E}_{[\alpha\beta]} = 0$ |
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- Strikingly similar when the setups are stationary in the observer's rest frame (since $F_{\alpha\beta;\gamma}U^{\gamma}$ and $\star F_{\alpha\beta;\gamma}U^{\gamma}$ vanish).

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- *Charges*: the gravitational analogue of $\rho_c$ is $2\rho_m + T^{\alpha}_{\ \alpha}$ ($\rho_m + 3p$ for a perfect fluid) $\Rightarrow$ in gravity, pressure and all material stresses contribute as sources.

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- *Ampére law*: in stationary (in the observer's rest frame) setups, equations $B_{[\alpha\beta]}$ and $\mathbb{H}_{[\alpha\beta]}$ match up to a factor of 2 $\Rightarrow$ currents of mass/energy source gravitomagnetism like currents of charge source magnetism.

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- *Absence of electromagnetic-like induction effects in gravity*:
  - $\mathbb{E}_{\mu\gamma}$ always symmetric $\Rightarrow$ no gravitational analogue to Faraday's law of induction!

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- *Absence of electromagnetic-like induction effects in gravity*:
  - Induction term $\star F_{\alpha\beta;\gamma}U^{\gamma}$ in $B_{[\alpha\beta]}$ has no counterpart in $\mathbb{H}_{[\alpha\beta]}$ $\Rightarrow$ no gravitational analogue to the magnetic fields induced by time varying electric fields.

# Magnetic dipole vs Gyroscope

| Electromagnetic Force on a Magnetic Dipole | Gravitational Force on a Spinning Particle |
|---|---|
| $F_{EM}^{\beta} = \frac{q}{2m} B^{\alpha\beta} S_{\alpha}$ | $F_{G}^{\beta} = -\mathbb{H}^{\alpha\beta} S_{\alpha}$ |

The explicit analogy between $F_{EM}^{\beta}$ and $F_{G}^{\beta}$ is ideally suited to:

- Compare the two interactions: amounts to compare $B_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$, which is crystal clear from the equations for tidal tensors:

| Magnetic Tidal Tensor | Gravito-Magnetic Tidal tensor |
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The explicit analogy between $F^{\beta}_{EM}$ and $F^{\beta}_{G}$ is ideally suited to:

- Compare the two interactions: amounts to compare $B_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$, which is crystal clear from the equations for tidal tensors:
  - Unveils similarities between the two forces which allow us visualize, in analogy with the more familiar electromagnetic ones, gravitational effects which are not transparent in the Papapetrou's original form.
  - and fundamental differences which prove especially enlightening to the understanding of spin-curvature coupling.

# Some Fundamental Differences

| Electromagnetic Force on a Magnetic Dipole | Gravitational Force on a Spinning Particle |
|---|---|
| $F^{\beta}_{EM} = \frac{q}{2m} B^{\alpha\beta} S_{\alpha}$ | $F^{\beta}_{G} = -\mathbb{H}^{\alpha\beta} S_{\alpha}$ |

- $B_{\alpha\beta}$ is linear, whereas $\mathbb{H}_{\alpha\beta}$ is *not*
- *In vacuum* $\mathbb{H}_{[\alpha\beta]} = 0$ (symmetric tensor);
- $B_{[\alpha\beta]} = \frac{1}{2} \star F_{\alpha\beta;\gamma} U^{\gamma} \neq 0$ (even in vacuum)
- $\mathbb{H}_{\alpha\beta} U^{\beta} = 0$ (spatial tensor) $\Rightarrow F^{\beta}_{G} U_{\beta} = 0$ (it is a spatial force).
- $B_{\alpha\beta} U^{\beta} \neq 0 \Rightarrow F^{\beta}_{EM} U_{\beta} \neq 0$ (non-vanishing time projection!)

# Symmetries of Tidal tensors

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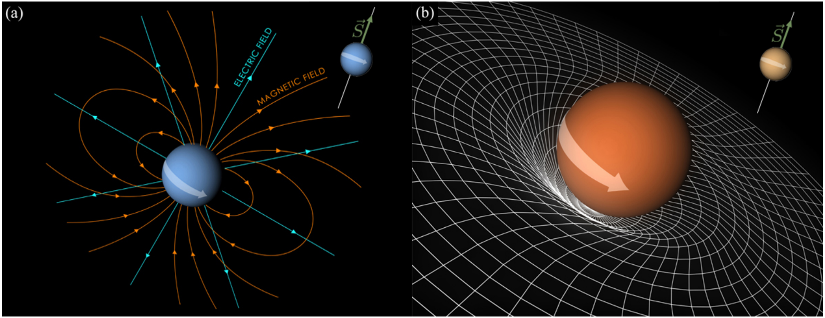
| --- | --- |
| $F^{\beta}_{EM} = \frac{q}{2m} B^{\alpha\beta} S_{\alpha}$ | $F^{\beta}_{G} = -\mathbb{H}_{\alpha}{}^{\beta} S^{\alpha}$ |
| **Magnetic Tidal Tensor** | **Gravito-magnetic Tidal Tensor** |
| $B_{[\alpha\beta]} = \frac{1}{2} \star F_{\alpha\beta;\gamma} U^{\gamma} - 2\pi\epsilon_{\alpha\beta\sigma\gamma} j^{\sigma} U^{\gamma}$ | $\mathbb{H}_{[\alpha\beta]} = -4\pi\epsilon_{\alpha\beta\sigma\gamma} J^{\sigma} U^{\gamma}$ |

- If the fields *do not vary* along the test particle's wordline, $\star F_{\alpha\beta;\gamma} U^{\gamma} = 0$ and the tidal tensors have the same symmetries.
- Allows for a similarity between the two interactions.

# Gravitational Spin-Spin Force

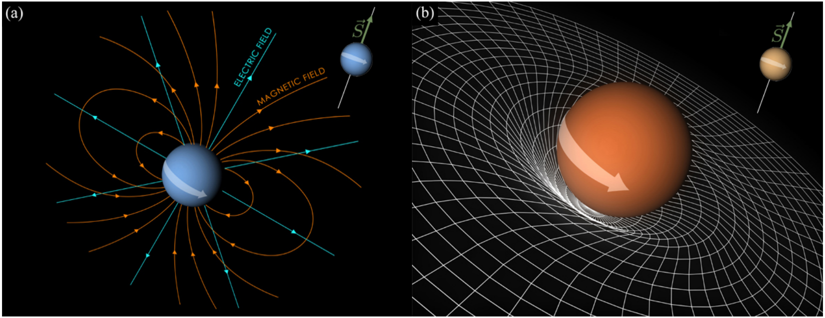



$$F_G^i \simeq \frac{3}{c}\left[\frac{(\vec{r}\cdot\vec{J})}{r^5}\delta^{ij} + 2\frac{r^{(i}J^{j)}}{r^5} - 5\frac{(\vec{r}\cdot\vec{J})r^i r^j}{r^7}\right] S_j \overset{J\leftrightarrow\mu}{=} F_{EM}^i$$

An analogy already known from linearized theory (Wald, 1972), and usually cast in the form:

$$\vec{F}_G = -\nabla(\vec{S}\cdot\vec{B}_G)$$

# Gravitational Spin-Spin Force

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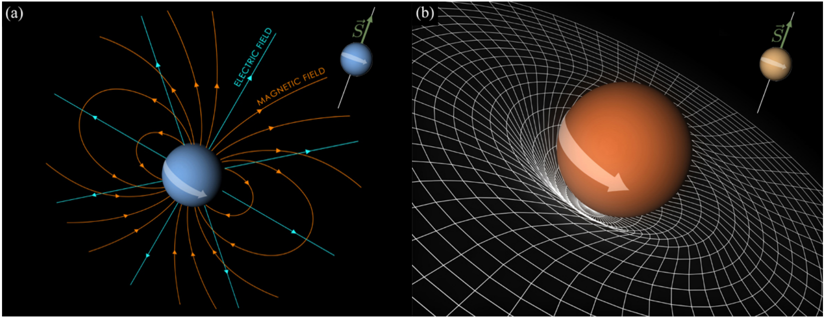
An analogy already known from linearized theory (Wald, 1972), and usually cast in the form:

$$\vec{F}_G = -\nabla(\vec{S}\cdot\vec{B}_G) + \frac{\partial}{\partial t}\left(\epsilon^{ijk}\phi_{,k}\right) S_i\vec{e}_j$$

- Holds only if the gyroscope is *at rest* and the fields are stationary.

# Gravitational Spin-Spin Force

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- Not suitable to describe motion; accounts only for spin-spin coupling.

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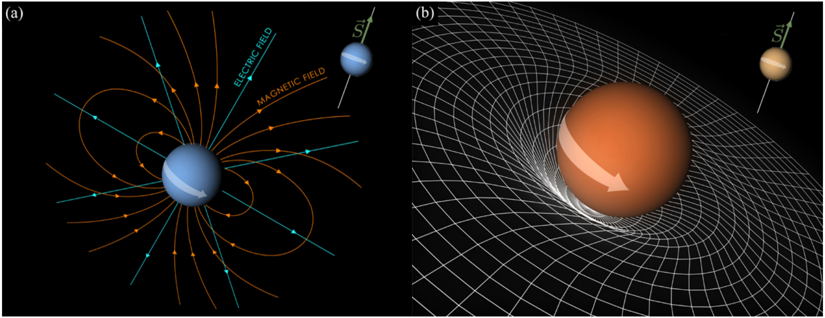


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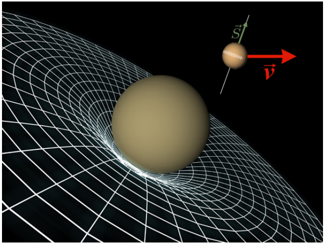
- Not suitable to describe motion; accounts only for spin-spin coupling.

The gyroscope deviates from geodesic motion even in the absence of rotating sources (e.g. Schwarzschild spacetime).



An effect readily visualized using the explicit analogy (always valid!!):

| Force on a Magnetic Dipole | Force on a Gyroscope |
|---|---|
| $F^{\beta}_{EM} = \frac{q}{2m} B^{\alpha\beta} S_{\alpha}$ | $F^{\beta}_{G} = -\mathbb{H}^{\alpha\beta} S_{\alpha}$ |

- It the magnetic tidal tensor, *as seen by the test particle*, that determines the force exerted upon it;
- Hence the gyroscope deviates from geodesic motion by the same reason that a magnetic dipole suffers a force even in the coulomb field of a point charge: in its "rest" frame, there is a non-vanishing magnetic tidal tensor.

# Symmetries of Tidal tensors

| Electromagnetic Force on a Magnetic Dipole | Gravitational Force on a Gyroscope |
|---|---|
| $F^{\beta}_{EM} = \frac{q}{2m} B^{\alpha\beta} S_{\alpha}$ | $F^{\beta}_{G} = -\mathbb{H}_{\alpha}{}^{\beta} S^{\alpha}$ |
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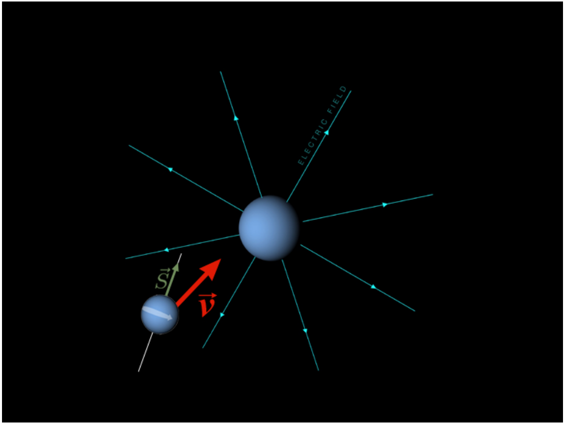
| $B_{[\alpha\beta]} = \frac{1}{2} \star F_{\alpha\beta;\gamma} U^{\gamma} - 2\pi \epsilon_{\alpha\beta\sigma\gamma} j^{\sigma} U^{\gamma}$ | $\mathbb{H}_{[\alpha\beta]} = -4\pi \epsilon_{\alpha\beta\sigma\gamma} J^{\sigma} U^{\gamma}$ |

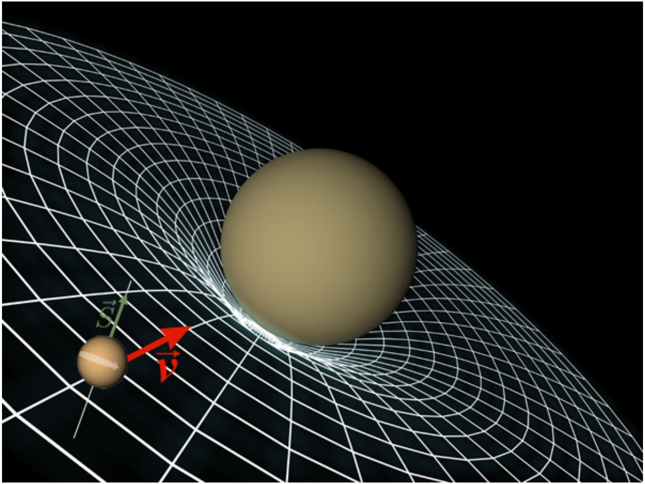
- If the fields *vary* along the test particle's wordline, the two interactions differ significantly.
- In vacuum, $\mathbb{H}_{[\alpha\beta]}$ is *always* symmetric, whereas $B_{\alpha\beta}$ is *not* : $B_{[\alpha\beta]} = \frac{1}{2} \star F_{\alpha\beta;\gamma} U^{\gamma}$.

# Radial Motion in Coulomb Field




- The dipole sees a time varying electric field;
- Thus, $B_{[\alpha\beta]} = \frac{1}{2} \star F_{\alpha\beta;\gamma} U^{\gamma} \neq 0$
  - $F^i_{EM} = \frac{q}{2m} B_{\alpha}{}^{i} S^{\alpha} = \gamma \frac{qQ}{2mr^3} (\vec{v} \times \vec{S})^i$

# Radial Motion in Schwarzschild




- No analogous gravitational effect: $F_G^\alpha = 0 \Rightarrow$ gyroscope moves along a geodesic.

# Scalar Invariants

The Riemann tensor (20 independent components) splits irreducibly into three spatial tensors (Louis Bel, 1958):

$$
\begin{aligned}
R_{\alpha\beta}^{\ \ \gamma\delta} \;=\;& 4\left\{2\tilde{U}_{[\alpha}\tilde{U}^{[\gamma} + g_{[\alpha}^{\ \ [\gamma}\right\}\tilde{\mathbb{E}}_{\beta]}^{\ \ \delta]} \\
&+2\left\{\epsilon_{\alpha\beta\mu\nu}\tilde{\mathbb{H}}^{\mu[\delta}\tilde{U}^{\gamma]}\tilde{U}^{\nu} + \epsilon^{\gamma\delta\mu\nu}\tilde{\mathbb{H}}_{\mu[\beta}\tilde{U}_{\alpha]}\tilde{U}_{\nu}\right\} \\
&+\epsilon_{\alpha\beta\mu\nu}\epsilon^{\gamma\delta\sigma\tau}\tilde{U}^{\mu}\tilde{U}_{\sigma}\left\{\tilde{\mathbb{F}}^{\nu}{}_{\tau} + \tilde{\mathbb{E}}^{\nu}{}_{\tau} - g^{\nu}_{\tau}\tilde{\mathbb{E}}^{\rho}{}_{\rho}\right\}
\end{aligned}
$$

$$
\mathbb{F}_{\alpha\beta} \equiv \star R \star_{\alpha\mu\beta\nu} U^{\mu}U^{\nu}
$$

- $\mathbb{E}_{\alpha\beta}$, $\mathbb{F}_{\alpha\beta}$: spatial, symmetric tensors $\Rightarrow$ 6 independent components each;
- $\mathbb{H}_{\alpha\beta}$: spatial, traceless tensor $\Rightarrow$ 8 independent components
- $\mathbb{F}_{\alpha\beta}$ has no electromagnetic analogue.

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& +2\left\{\epsilon_{\alpha\beta\mu\nu}\tilde{\mathbb{H}}^{\mu[\delta}\tilde{U}^{\gamma]}\tilde{U}^{\nu} + \epsilon^{\gamma\delta\mu\nu}\tilde{\mathbb{H}}_{\mu[\beta}\tilde{U}_{\alpha]}\tilde{U}_{\nu}\right\} \\
& +\epsilon_{\alpha\beta\mu\nu}\epsilon^{\gamma\delta\sigma\tau}\tilde{U}^{\mu}\tilde{U}_{\sigma}\left\{\tilde{\mathbb{F}}^{\nu}{}_{\tau} + \tilde{\mathbb{E}}^{\nu}{}_{\tau} - g_{\tau}^{\nu}\tilde{\mathbb{E}}^{\rho}{}_{\rho}\right\}
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\end{aligned}
$$

- $\mathbb{E}_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$ completely encode the 14 (6+8) independent components of the Riemann tensor in vacuum (Weyl Tensor).

# Scalar Invariants

Although each of these spatial tensors is determined by the 4-velocity $U^\alpha$ of the observer measuring it:

$$\begin{aligned} \mathbb{E}_{\alpha\beta} &\equiv R_{\alpha\mu\beta\nu} U^\mu U^\nu \\ \mathbb{H}_{\alpha\beta} &\equiv \star R_{\alpha\mu\beta\nu} U^\mu U^\nu \end{aligned}$$

it can be shown that the following expressions are observer independent (in vacuum)::

$$\begin{aligned} \mathbb{E}^{\alpha\gamma}\mathbb{E}_{\alpha\gamma} - \mathbb{H}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma} &= \frac{1}{8} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \\ \mathbb{E}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma} &= \frac{1}{16} R_{\alpha\beta\gamma\delta} \star R^{\alpha\beta\gamma\delta} \end{aligned}$$

- gravitational tidal tensors form scalar invariants!

# Scalar Invariants

- in vacuum

$$\mathbb{E}^{\alpha\gamma}\mathbb{E}_{\alpha\gamma} - \mathbb{H}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma} = \frac{1}{8}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$$

$$\mathbb{E}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma} = \frac{1}{16}R_{\alpha\beta\gamma\delta} \star R^{\alpha\beta\gamma\delta}$$

- *Formally* analogous to the electromagnetic scalar invariants:

$$\vec{E}^2 - \vec{B}^2 = -\frac{1}{2}F_{\alpha\beta}F^{\alpha\beta}$$

$$\vec{E}.\vec{B} = -\frac{1}{4}F_{\alpha\beta} \star F^{\alpha\beta}$$

# Scalar Invariants

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- **This is a purely formal analogy,** relating electromagnetic *fields* with gravitational *tidal tensors* (which are one order higher in differentiation!)

# Scalar Invariants – Electromagnetism

$$
\begin{aligned}
\vec{E}^2 - \vec{B}^2 &= -\frac{1}{2} F_{\alpha\beta} F^{\alpha\beta} \\
\vec{E}.\vec{B} &= -\frac{1}{4} F_{\alpha\beta} \star F^{\alpha\beta}
\end{aligned}
$$

- $\vec{E}.\vec{B} = 0$ and $\vec{E}^2 - \vec{B}^2 > 0 \Rightarrow$ there are observers for which the magnetic field $\vec{B}$ vanishes.
- $\vec{E}.\vec{B} = 0$ and $\vec{E}^2 - \vec{B}^2 < 0 \Rightarrow$ there are observers for which the electric field $\vec{E}$ vanishes.

# Scalar Invariants – Electromagnetism

$$\vec{E}^2 - \vec{B}^2 = -\frac{1}{2} F_{\alpha\beta} F^{\alpha\beta}$$
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- $\vec{E}.\vec{B} = 0$ and $\vec{E}^2 - \vec{B}^2 < 0 \Rightarrow$ there are observers for which the electric field $\vec{E}$ vanishes.
- $\vec{E}^2 - \vec{B}^2$ and $\vec{E}.\vec{B}$ are the only algebraically independent invariants one can define from the Maxwell tensor $F^{\alpha\beta}$.

# Scalar Invariants – Gravity (Vacuum)

In vacuum, one can construct 4 independent scalar invariants from Riemann tensor (would be 14 in general):

$$\mathbb{E}^{\alpha\gamma}\mathbb{E}_{\alpha\gamma} - \mathbb{H}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma} = \frac{1}{8}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \equiv \frac{1}{8}\mathbf{R}.\mathbf{R}$$

$$\mathbb{E}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma} = \frac{1}{16}R_{\alpha\beta\gamma\delta} \star R^{\alpha\beta\gamma\delta} \equiv \frac{1}{16}\mathbf{R}.^{\star}\mathbf{R}$$

$$\mathbb{E}^{\alpha}{}_{\beta}\mathbb{E}^{\beta}{}_{\gamma}\mathbb{E}^{\gamma}{}_{\alpha} - 3\mathbb{E}^{\alpha}{}_{\beta}\mathbb{H}^{\beta}{}_{\gamma}\mathbb{H}^{\gamma}{}_{\alpha} = \frac{1}{16}R^{\alpha\beta}{}_{\lambda\mu}R^{\lambda\mu}{}_{\rho\sigma}R^{\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta} \equiv A$$

$$\mathbb{H}^{\alpha}{}_{\beta}\mathbb{H}^{\beta}{}_{\gamma}\mathbb{H}^{\gamma}{}_{\alpha} - 3\mathbb{E}^{\alpha}{}_{\beta}\mathbb{E}^{\beta}{}_{\gamma}\mathbb{H}^{\gamma}{}_{\alpha} = \frac{1}{16}R^{\alpha\beta}{}_{\lambda\mu}R^{\lambda\mu}{}_{\rho\sigma}R^{\rho\sigma} \star R^{\rho\sigma}{}_{\alpha\beta} \equiv B$$

- $\mathbf{R}.^{\star}\mathbf{R} = 0$ and $\mathbf{R}.\mathbf{R} \neq 0$ is not sufficient to ensure that there are observers for which $\mathbb{H}_{\alpha\gamma}$ (or $\mathbb{E}_{\alpha\gamma}$) vanishes.
- Needs also $M \equiv I^3/J^2 - 6$ to be real positive, where

$$I \equiv \frac{1}{8}\mathbf{R}\cdot\mathbf{R} + \frac{i}{8}\star\mathbf{R}\cdot\mathbf{R}; \quad J \equiv A - iB$$

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$$\begin{aligned} \mathbb{E}^{\alpha\gamma}\mathbb{E}_{\alpha\gamma} - \mathbb{H}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma} &= \frac{1}{8} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \equiv \frac{1}{8}\mathbf{R}.\mathbf{R} \\ \mathbb{E}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma} &= \frac{1}{16} R_{\alpha\beta\gamma\delta} \star R^{\alpha\beta\gamma\delta} \equiv \frac{1}{16}\mathbf{R}.{}^{\star}\mathbf{R} \end{aligned}$$

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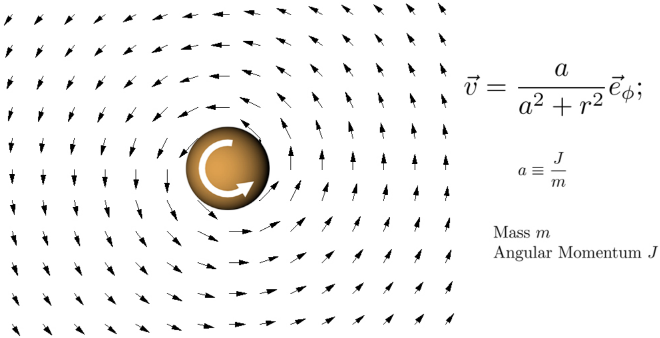
- $M \geq 0$ (real), $\mathbf{R}.{}^{\star}\mathbf{R} = 0$ and $\mathbf{R}.\mathbf{R} > 0 \Rightarrow$ there are observers for which $\mathbb{H}_{\alpha\gamma}$ vanishes ("Purely Electric" spacetime) .
- $M \geq 0$ (real), $\mathbf{R}.{}^{\star}\mathbf{R} = 0$ and $\mathbf{R}.\mathbf{R} < 0 \Rightarrow$ there *would* be observers for which $\mathbb{E}_{\alpha\gamma}$ vanishes (but there are no known "Purely Magnetic" vacuum solutions; conjecture: do not exist)

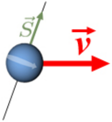
Kerr metric:
$$\begin{cases} \mathbb{E}^{\alpha\gamma}\mathbb{E}_{\alpha\gamma} - \mathbb{H}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma} \approx \dfrac{6m^2}{r^6} > 0; \quad I^3 = 6J^2 \text{ (Petrov D)} \\[2ex] \mathbb{E}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma} \approx \dfrac{18Jm\cos\theta}{r^7} = 0 \text{ in the plane } \theta = \pi/2 \end{cases}$$

- In the equatorial plane $\theta = \pi/2$, there are observers for which **locally** $\mathbb{H}_{\alpha\gamma} = 0$ :




- Observers such that $v^\phi \equiv \dfrac{U^\phi}{U^t} = \dfrac{a^2}{a^2 + r^2}$ (**not** the "co-rotating" observers!)

- Impossible in the electromagnetic analog.

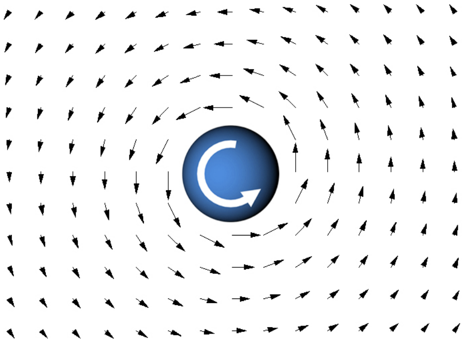



- A moving dipole sees a time-varying electromagnetic field;
- Thus $B_{\alpha\beta}$ must be non vanishing (consequence of $\nabla \times \vec{B} = \partial \vec{E} / \partial t$):

$$B_{[\alpha\beta]} = \frac{1}{2} \star F_{\alpha\beta;\gamma} U^{\gamma} \neq 0 \Rightarrow B_{\alpha\beta} \neq 0$$

Spinning Charge: $\begin{cases} \vec{E}^2 - \vec{B}^2 = \dfrac{q^2}{r^4} - \dfrac{\mu^2(5+3\cos 2\theta)}{2r^6} > 0 \\ \vec{E}.\vec{B} = \dfrac{2\mu q \cos\theta}{r^5} = 0 \text{ in the plane } \theta = \pi/2 \end{cases}$

- In the equatorial plane $\theta = \pi/2$, there are observers for which **locally** $\vec{B} = 0$:

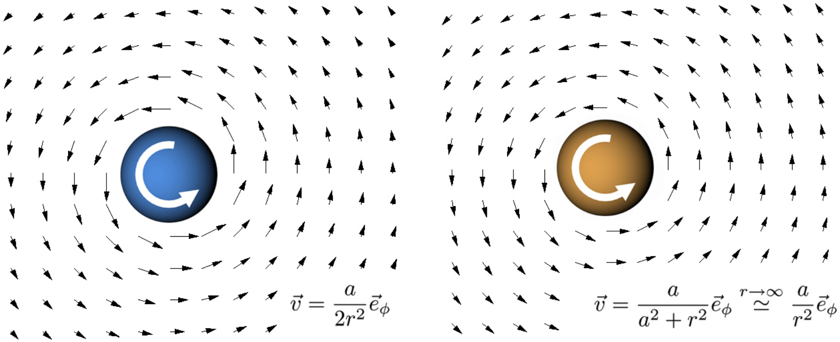


$$\vec{v} = \frac{J}{2mr^2}\vec{e}_\phi$$

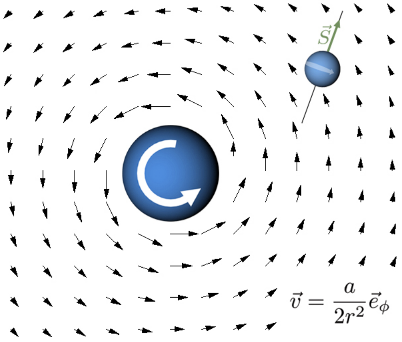
Charge $q$, Mass $m$
Dipole Moment $\mu$
Angular Momentum $J$

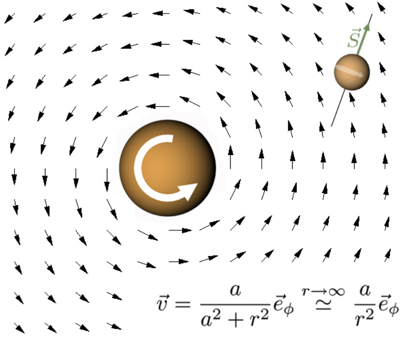
- Observers such that $v^\phi \equiv \dfrac{U^\phi}{U^t} = \dfrac{J}{2mr^2}$ (**not** the "co-rotating" observers!)

# Spinning Charge vs Spinning mass




- For $r \to \infty$, the two velocities asymptotically match! (up to a factor of 2)




- No precession:

$$\vec{B} = 0 \Rightarrow \frac{d\vec{S}}{dt} = 0$$

- There is a Force applied:

$$B_{\alpha\beta} \neq 0 \Rightarrow F^{\alpha}_{EM} = \frac{q}{2m} B^{\beta\alpha} S_{\beta} \neq 0$$

(consequence of $\nabla \times \vec{B} = \partial\vec{E}/\partial t$)

- Gyroscope precesses:

$$\frac{d\vec{S}}{dt} \neq 0$$

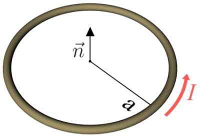
- No Force on Gyroscope:

$$\mathbb{H}_{\alpha\beta} = 0 \Rightarrow F^{\alpha}_{G} = -\mathbb{H}^{\beta\alpha} S_{\beta} = 0$$

Time projection of $F^{\alpha}_{EM}$ in the dipole's proper frame:

$$F^{\alpha}_{EM}U_{\alpha} = -B^{i0}\mu_i = \frac{\partial \vec{B}}{\partial t}.\vec{\mu}$$

- The magnetic dipole may be thought as a small current loop.




(Area of the loop

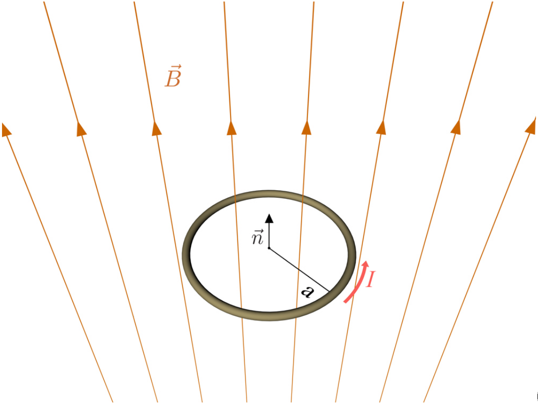
$A = 4\pi a^2$; $I \equiv$ current through the loop, $\vec{n} \equiv$unit vector normal to the loop)

- The magnetic dipole moment is given by $\vec{\mu} = IA\vec{n}$

Time projection of $F^{\alpha}_{EM}$ in the dipole's proper frame:

$$F^{\alpha}_{EM}U_{\alpha} = -B^{i0}\mu_i = \frac{\partial \vec{B}}{\partial t}.\vec{\mu} = \frac{\partial \vec{B}}{\partial t}.\vec{n}AI = \frac{\partial \Phi}{\partial t}I$$

- The magnetic dipole may be thought as a small current loop.



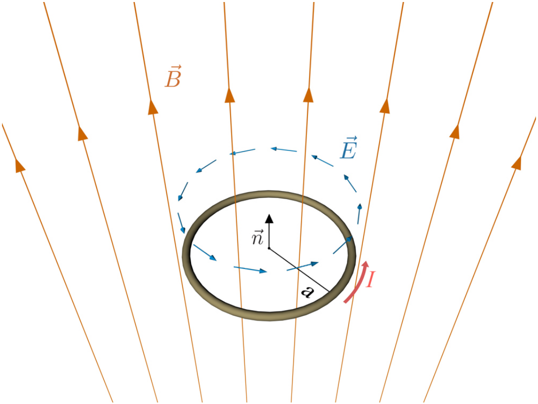

(Area of the loop
$A = 4\pi a^2$; $I \equiv$ current through the loop, $\vec{n} \equiv$unit vector normal to the loop)

- $\vec{B}A\vec{n} = \Phi \equiv$ magnetic flux trough the loop

Therefore, by Faraday´s law of induction:

$$F^{\alpha}_{EM}U_{\alpha} = -B^{i0}\mu_i = \frac{\partial \vec{B}}{\partial t}.\vec{\mu} = \frac{\partial \vec{B}}{\partial t}.\vec{n}AI = \frac{\partial \Phi}{\partial t}I = -I\oint_{loop}\vec{E}.\vec{ds}$$

- $\vec{E} \equiv$ Induced electric field




- Hence $F^{\alpha}_{EM}U_{\alpha}$ is minus the power transferred to the dipole by Faraday's law of induction.

- $F^{\alpha}_{EM} U_{\alpha} = DE/d\tau$ is minus the power transferred to the dipole by Faraday's law of induction;
  - yields the variation of the energy $E$ as measured in the dipole's center of mass rest frame;
  - is reflected in a variation of the dipole's proper mass $m = -P^{\alpha} U_{\alpha}$

# Time Projection of $F_G^\alpha$ — no gravitational induction

Since $\mathbb{H}_{\alpha\beta}$ is a spatial tensor, we *always have*

$$F_G^\alpha U_\alpha = -\frac{dm}{d\tau} = 0$$

- No work is done by induction $\Rightarrow$ the energy of the gyroscope, *as measured in its center of mass frame,* is constant;
  - the proper mass $m = -P^\alpha U_\alpha$ of the gyroscope is constant.
- Spatial character of gravitational tidal tensors <u>precludes</u> induction effects analogous to the electromagnetic ones.

## Time Projection of $F_G^\alpha$ — no gravitational induction

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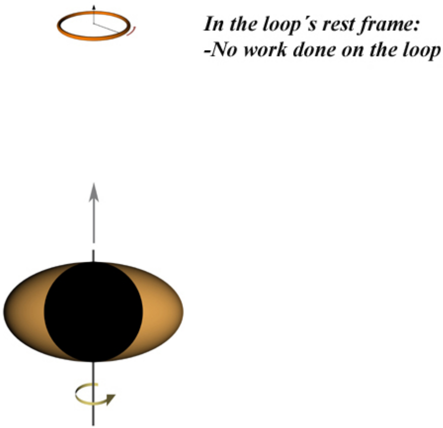

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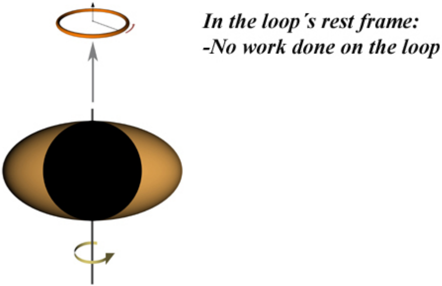

# Time components on arbitrary frames

- Electromagnetism:

In an arbitrary frame, in which the dipole has 4-velocity $U^\beta = \gamma(1, \vec{v})$, the time component of the force exerted on a magnetic dipole is:

$$
\begin{aligned}
(F_{EM})_0 &\equiv -\frac{DE}{d\tau} = \frac{F^\beta_{EM} U_\beta}{\gamma} - F^i_{EM} v_i = -\left(\frac{1}{\gamma}\frac{dm}{d\tau} + F^i_{EM} v_i\right) \\
&\equiv -(\mathcal{P}_{mech} + \mathcal{P}_{ind})
\end{aligned}
$$

where $E \equiv -P_0$ is the energy of the dipole and we identify:

- $\mathcal{P}_{ind} = \frac{1}{\gamma}\frac{dm}{d\tau} = -F^\beta_{EM} U_\beta / \gamma \equiv$ induced power
- $\mathcal{P}_{mech} = F^i_{EM} v_i$ "mechanical" power transferred to the dipole by the 3-force $F^i_{EM}$ exerted upon it.

# Static Observers — Electromagnetism

$$(F_{EM})_0 = \frac{q}{2m} B_{\alpha 0} S^{\alpha} = \star F_{\alpha\gamma;0} U^{\gamma} S^{\alpha} = 0$$

- no work is done on the magnetic dipole.
  - Related to a basic principle from electromagnetism: the total amount of **work done by a static magnetic field on an arbitrary system of currents is zero.**

$$\vec{F} = q\vec{v} \times \vec{B} \;\Rightarrow\; \vec{F} \perp \vec{v}$$

(Lorentz Force on *each* individual charge)

# Static Observers — Electromagnetism

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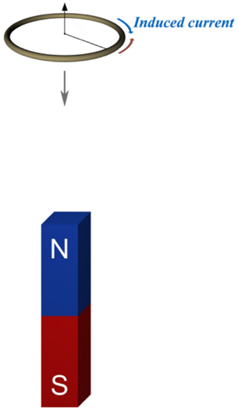

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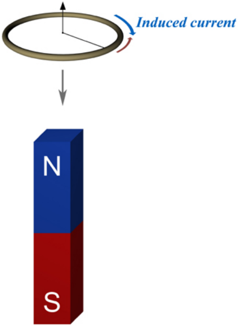


# Static Observers — Gravity

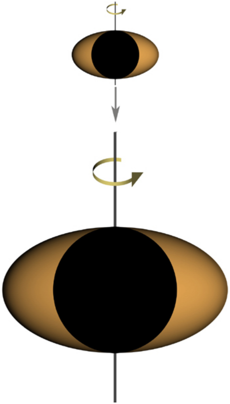
- In gravity, since those induction effects are absent, such cancellation does not occur:

$$(F_G)_0 = -\frac{DE}{d\tau} = -F_G^i v_i \neq 0$$

- Therefore, the stationary observer must measure a non-zero work done on the gyroscope.
  - That is to say, **a static "gravitomagnetic field" (unlike its electromagnetic counterpart) does work.**
  - And there is a known consequence of this fact: the spin dependent upper bound for the energy released when two black holes collide, obtained by Hawking (1971) from the area law.
  - For the case with spins aligned, from Hawking's expression one can infer a gravitational spin-spin interaction energy (Wald, 1972).

## Static Observers — Gravitational spin interaction

Take the gyroscope to be a small Kerr black hole of spin $S \equiv \sqrt{S^\alpha S_\alpha}$ falling along the symmetry axis of a larger Kerr black hole of mass $m$ and angular momentum $J = am$.



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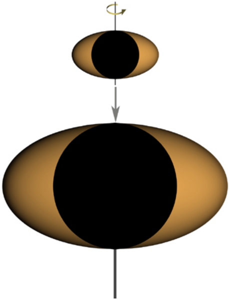

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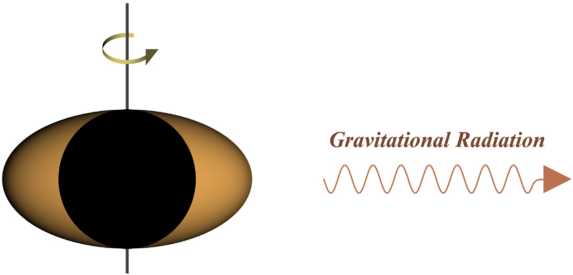

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- The time component of the force acting on the small black hole is given by:

$$(F_G)_0 \equiv \frac{DP_0}{D\tau} = -\frac{dE}{d\tau} = -\frac{2ma\left(3r^2 - a^2\right) U^r S}{\left(r^2 + a^2\right)^3}$$

  Integrating this equation from infinity to the horizon one obtains

$$\int_\infty^{r_+} (F_G)_0 \equiv \Delta E = \frac{aS}{2m\left[m + \sqrt{m^2 - a^2}\right]} ,$$

  which is precisely Hawking's spin-spin interaction energy for this particular setup.

# Conclusions

- The tidal tensor formalism unveils an exact, fully general analogy between the force on a gyroscope and on a magnetic dipole;
- at the same time it makes transparent both the similarities and key differences between the two interactions;
- The non-geodesic motion of a spinning test particle not only can be easily understood, but also exactly described, by a simple application of this analogy.
- This analogy sheds light on important aspects of spin-curvature coupling;
  - the fact that the mass of a gyroscope is constant (as signaling the absence of gravitational effects analogous to electromagnetic induction);
  - namely, Hawking's spin dependent upper bound for the energy released on black hole collision (as arising from the fact that gravitomagnetic fields *do work*);
- Issues concerning previous approaches in the literature were clarified — namely, the limit of validity of the usual linear gravito-electromagnetic analogy, and the physical interpretation of the magnetic parts of the Riemann/Weyl tensors.

# Acknowledgments

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Thank you