Motion of gyroscopes around Schwarzschild and Kerr BH – exact gravito-electromagnetic analogies

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Introduction

- It is known, since the works of Mathisson and Papapetrou that spinning particles follow worldlines which are not geodesics;
- In linearized theory, the gravitational force exerted on a spinning pole-dipole test particle (hereafter a gyroscope) takes a form: \( \vec{F}_G = \nabla (\vec{B}_G \cdot \vec{S}) \) similar to the electromagnetic force on a magnetic dipole (Wald 1972).
  - But only if the gyroscope is at “rest” in a stationary, weak field!
- This analogy may be cast in an exact form (Natário, 2007) using the “Quasi-Maxwell” formalism, which holds if the gyroscope’s 4-velocity is a Killing vector of a stationary spacetime.
Introduction

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  - But only if the gyroscope is at “rest” in a stationary, weak field!
- This analogy may be cast in an exact form (Natário, 2007) using the “Quasi-Maxwell” formalism, which holds if the gyroscope’s 4-velocity is a Killing vector of a stationary spacetime.
- There is an exact, covariant and fully general analogy relating the two forces, which is made explicit in the tidal tensor formalism (Costa & Herdeiro 2008).
- We will exemplify how this analogy provides new intuition for the understanding of spin curvature coupling.
Force on Magnetic Dipole

\[ F_{EM}^{\alpha} = \frac{DP^{\alpha}}{d\tau} = \frac{1}{2} F_{\mu\nu}^{\alpha} Q_{\mu\nu} \]

- \( F_{\mu\nu} \equiv \) Maxwell tensor
- \( Q_{\mu\nu} \equiv \) dipole moment tensor

\[
Q_{\mu\nu} = \begin{bmatrix}
0 & -d^x & -d^y & -d^z \\
-d^x & 0 & \mu^z & -\mu^y \\
-d^y & -\mu^z & 0 & \mu^x \\
-d^z & \mu^y & -\mu^x & 0
\end{bmatrix}
\]
For a magnetic dipole ($\vec{d} = 0$):

\[
Q_{\mu\nu} = \begin{bmatrix}
0 & -d^x & -d^y & -d^z \\
d^x & 0 & \mu^z & -\mu^y \\
d^y & -\mu^z & 0 & \mu^x \\
d^z & \mu^y & -\mu^x & 0
\end{bmatrix}
\]

In Relativity, electric and magnetic dipole moments do not exist as independent entities;

- $\vec{d}$ and $\vec{\mu}$ are the time and space components of the dipole moment 2-Form.
Force on Magnetic Dipole

\[ F_{EM}^\alpha = \frac{DP^\alpha}{d\tau} = \frac{1}{2} F_{\mu\nu}^{;\alpha} Q^{\mu\nu} \]

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For a magnetic dipole (\( \vec{d} = 0 \) in its proper frame):

\[
Q^{\mu\nu} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & \mu^z & -\mu^y \\
0 & -\mu^z & 0 & \mu^x \\
0 & \mu^y & -\mu^x & 0
\end{bmatrix} = \sigma S^{\mu\nu}
\]
Force on Magnetic Dipole

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- \( S^{\mu\nu} \equiv \) Spin tensor;
- \( \sigma \equiv \) gyromagnetic ratio (\( = q/2m \) for classical spin)

\[ F_{EM}^\alpha = \frac{DP^\alpha}{d\tau} = \frac{1}{2} \sigma F_{\mu\nu}^{\;\alpha} S^{\mu\nu} \]
Force on Magnetic Dipole

\[ F_{EM}^\alpha = \frac{DP^\alpha}{d\tau} = \frac{1}{2} \sigma F_{\mu\nu}^\alpha S^{\mu\nu} \]

- If Pirani supplementary condition \( S^{\mu\nu} U_\nu = 0 \) holds, then
  \[ S^{\mu\nu} = \epsilon^{\mu\nu\tau\lambda} S_\tau U_\lambda \]
- \( S^\alpha \equiv \) spin 4-vector; defined as the vector that, in the particle's proper frame, \( S^\alpha = (0, \vec{S}) \)
Force on Magnetic Dipole

\[ F_{EM}^\alpha = \frac{DP^\alpha}{d\tau} = \frac{1}{2} \sigma F_{\mu\nu}^{\ ;\alpha} S^{\mu\nu} \]

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\[ F_{EM}^\alpha = \frac{DP^\alpha}{d\tau} = \sigma \epsilon_{\mu\nu}^{\ \tau\lambda} F_{\mu\nu}^{\ ;\alpha} U_\lambda S_\tau = \sigma B^\alpha_\beta S^\beta \]
Force on Magnetic Dipole

\[ F_{EM}^{\alpha} = \frac{DP^{\alpha}}{d\tau} = \frac{1}{2} \sigma F_{\mu\nu;\alpha} S^{\mu\nu} \]

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\[ F_{EM}^{\alpha} = \frac{DP^{\alpha}}{d\tau} = \sigma \epsilon^{\mu\nu\tau\lambda} F_{\mu\nu;\alpha} U_\lambda S_\tau = \sigma B_\beta^{\alpha} S^\beta \]

- \( B_{\alpha\beta} \equiv \star F_{\alpha\gamma;\beta} U^\gamma \equiv \) magnetic tidal tensor
  - Measures the tidal effects produced by the magnetic field \( B^\alpha = \star F_\gamma^{\alpha} U^\gamma \) seen by the dipole of 4-velocity \( U^\gamma \).
Force on Magnetic Dipole

\[ F_{EM}^\alpha = \sigma B_\beta^\alpha S^\beta \]

- Covariant generalization of the usual 3-D expression (valid only in a frame where the dipole is at rest!):

\[ \vec{F}_{EM} = \nabla (\vec{\mu} \cdot \vec{B}) \]

- Yields the force exerted on a magnetic dipole moving with arbitrary velocity.
Force on Gyroscope

Papapetrou equation:

\[ F_G^{\alpha} \equiv \frac{DP^\alpha}{D\tau} = -\frac{1}{2} R_{\beta\mu\nu}^{\alpha} U^\beta S^{\mu\nu} \]
Force on Gyroscope

Papapetrou equation:

\[ F_G^\alpha \equiv \frac{DP^\alpha}{D\tau} = -\frac{1}{2} R^\alpha_{\beta\mu\nu} U^\beta S^\mu\nu \]

- If Pirani supplementary condition \( S^{\mu\nu} U_\nu = 0 \) holds, then
  \[ S^{\mu\nu} = \epsilon^{\mu\nu\tau\lambda} S_\tau U_\lambda \]

\[ \frac{DP^\alpha}{D\tau} = \frac{1}{2} \epsilon_{\mu\nu}^{\tau\lambda} R^{\mu\nu\alpha\beta} U_\lambda U_\beta S_\tau = -\mathbb{H}^\alpha_\beta S^\beta \]

- \( \mathbb{H}_{\alpha\beta} \equiv ”\text{Magnetic part of the Riemann tensor”}” \)
Magnetic-type Tidal Tensors

The electromagnetic force exerted on a magnetic dipole and the gravitational force causing the non-geodesic motion of a spinning test particle are analogous tidal effects:

- **Electromagnetic Force on a Magnetic Dipole**
  (Covariant form for $\vec{F}_{EM} = \nabla (\vec{\mu} \cdot \vec{B})$)

\[
F^\alpha_{EM} \equiv \frac{DP^\alpha}{D\tau} = \sigma B^\alpha_\gamma S^\gamma, \quad B^\alpha_\gamma \equiv \star F^\alpha_{\beta;\gamma} U^\beta
\]

- **Gravitational Force on a Gyroscope**
  (Papapetrou-Pirani equation)

\[
F^\alpha_G \equiv \frac{DP^\alpha}{D\tau} = -\mathbb{H}^\alpha_\gamma S^\gamma, \quad \mathbb{H}^\alpha_\gamma \equiv \star R^\alpha_{\beta;\gamma\sigma} U^\beta U^\sigma
\]

- Suggests the physical analogy: $B_{\alpha\beta} \leftrightarrow \mathbb{H}_{\alpha\beta}$
  - $B_{\alpha\beta} \equiv \text{magnetic tidal tensor}$;
  - $\mathbb{H}_{\alpha\beta} \equiv \text{gravito-magnetic tidal tensor}$. 
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The electromagnetic force exerted on a magnetic dipole and the gravitational force causing the non-geodesic motion of a spinning test particle are analogous tidal effects:

- **Electromagnetic Force on a Magnetic Dipole**
  (Covariant form for $\vec{F}_{EM} = \nabla (\vec{\mu} \cdot \vec{B})$)

  $$F_{EM}^\alpha \equiv \frac{DP^\alpha}{DT} = \sigma B_\gamma^\alpha S^\gamma, \quad B_\gamma^\alpha \equiv \star F_\beta^\alpha;_\gamma U^\beta$$

- **Gravitational Force on a Gyroscope**
  (Papapetrou-Pirani equation)

  $$F_G^\alpha \equiv \frac{DP^\alpha}{DT} = -\mathbb{H}_\gamma^\alpha S^\gamma, \quad \mathbb{H}_\gamma^\alpha \equiv \star R_\beta^\alpha;_{\gamma\sigma} U^\beta U^\sigma$$

$\sigma = \mu / S \equiv$ gyromagnetic ratio $\Rightarrow$ equals 1 for gravity

$\Rightarrow \vec{\mu} \leftrightarrow \vec{S}$
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The electromagnetic force exerted on a magnetic dipole and the gravitational force causing the non-geodesic motion of a spinning test particle are analogous tidal effects:

- **Electromagnetic Force on a Magnetic Dipole**
  (Covariant form for $\mathbf{\tilde{F}}_{EM} = \nabla(\mathbf{\tilde{\mu}}.\mathbf{\tilde{B}})$)

  $$F_{EM}^\alpha \equiv \frac{DP^\alpha}{DT} = \sigma B_\gamma^\alpha S^\gamma, \quad B_\gamma^\alpha \equiv \star F_\beta^\alpha;_\gamma U^\beta$$

- **Gravitational Force on a Gyroscope**
  (Papapetrou-Pirani equation)

  $$F_G^\alpha \equiv \frac{DP^\alpha}{DT} = -\mathcal{H}_\gamma^\alpha S^\gamma, \quad \mathcal{H}_\gamma^\alpha \equiv \star R_\beta^\alpha_{\gamma\sigma} U^\beta U^\sigma$$

  ▶ Relative minus sign: mass/charges of the same sign attract/repel one another ⇒ antiparallel charge/mass currents repel/attract.
### Magnetic Tidal Tensor

**Antisymmetric part:**

\[ B_{[\alpha \beta]} = \frac{1}{2} \star F_{\alpha \beta; \gamma} U^\gamma - 2\pi \epsilon_{\alpha \beta \sigma \gamma} j^\sigma U^\gamma \]

- **Covariant form for**

\[ \nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + 4\pi \vec{j} \]

**Trace:**

\[ B^\alpha_\alpha = 0 \]

- **Covariant form for**

\[ \nabla \cdot \vec{B} = 0 \]

### Gravito-Magnetic Tidal Tensor

**Antisymmetric part:**

\[ H_{[\alpha \beta]} = -4\pi \epsilon_{\alpha \beta \sigma \gamma} j^\sigma U^\gamma \]

**Trace:**

\[ H^\alpha_\alpha = 0 \]
<table>
<thead>
<tr>
<th>Magnetic Tidal Tensor</th>
<th>Gravito-Magnetic Tidal Tensor</th>
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<tr>
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<td>$B_{[\alpha\beta]} = \frac{1}{2} \star F_{\alpha\beta;\gamma} U^\gamma - 2\pi \epsilon_{\alpha\beta\sigma\gamma} j^\sigma U^\gamma$</td>
<td>$H_{[\alpha\beta]} = -4\pi \epsilon_{\alpha\beta\sigma\gamma} j^\sigma U^\gamma$</td>
</tr>
<tr>
<td><strong>Space projection</strong></td>
<td><strong>Time-Space projection</strong></td>
</tr>
<tr>
<td>of Maxwell equations:</td>
<td>of Einstein equations:</td>
</tr>
<tr>
<td>$F^{\alpha\beta}<em>{\ ;\beta} = J</em>\beta$</td>
<td>$R_{\mu\nu} = 8\pi \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\alpha_\alpha \right)$</td>
</tr>
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<td><strong>Trace:</strong></td>
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Electric-type Tidal Tensors

Electric-type tidal forces are described in an invariant way through the wordline deviation equations:

- **Electromagnetic**

\[ \frac{D^2 \delta x^\alpha}{D\tau^2} = \frac{q}{m} E^\alpha_\gamma \delta x^\gamma, \quad E^\alpha_\gamma \equiv F^\alpha_{\beta;\gamma} U^\beta \]

- **Gravitational (geodesic deviation)**

\[ \frac{D^2 \delta x^\alpha}{D\tau^2} = -\mathbb{E}^\alpha_\gamma \delta x^\gamma, \quad \mathbb{E}^\alpha_\gamma \equiv R^\alpha_{\beta\gamma\sigma} U^\beta U^\sigma \]

which yield the acceleration of the vector \( \delta x^\alpha \) connecting two particles with the same \((\text{Ciufolini, 1986})\) 4-velocity \( U^\alpha \) — and the same \( q/m \) ratio in the electromagnetic case.

(Notation: \( F_{\alpha\beta} \equiv \text{Maxwell tensor} \), \( R_{\alpha\beta\gamma\sigma} \equiv \text{Riemann tensor} \))
Electric-type Tidal Tensors

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- **Gravitational (geodesic deviation)**

\[ \frac{D^2 \delta x^\alpha}{D \tau^2} = -E^\alpha_\gamma \delta x^\gamma, \quad E^\alpha_\gamma \equiv R^\alpha_\beta\gamma\sigma U^\beta U^\sigma \]

- Suggests the physical analogy: \( E_{\alpha\beta} \longleftrightarrow E_{\alpha\beta} \)

- \( E_{\alpha\beta} \) is the covariant derivative of the electric field \( E^\alpha = F^{\alpha\mu} U_\mu \) measured by the observer with (fixed) 4-velocity \( U^\alpha \);
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  \[
  \frac{D^2 \delta x^\alpha}{D\tau^2} = \frac{q}{m} E^\alpha_\gamma \delta x^\gamma, \quad E^\alpha_\gamma \equiv F^\alpha_{\beta;\gamma} U^\beta
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- **Gravitational (geodesic deviation)**

  \[
  \frac{D^2 \delta x^\alpha}{D\tau^2} = -\mathbb{E}^\alpha_\gamma \delta x^\gamma, \quad \mathbb{E}^\alpha_\gamma \equiv R^\alpha_{\beta\gamma\sigma} U^\beta U^\sigma
  \]

- Suggests the physical analogy: \( E_{\alpha\beta} \leftrightarrow \mathbb{E}_{\alpha\beta} \)

- Hence:

  \( E_{\alpha\beta} \equiv \text{electric tidal tensor}; \quad \mathbb{E}_{\alpha\beta} \equiv \text{gravito-electric tidal tensor}. \)
**Analogy based on tidal tensors (Costa-Herdeiro 2008)**

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<td><strong>Worldline deviation:</strong></td>
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<td>( \frac{D^2 \delta x^\alpha}{D \tau^2} = \frac{q}{m} E^\alpha_\beta \delta x^\beta )</td>
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<td><strong>Force on magnetic dipole:</strong></td>
<td><strong>Force on gyroscope:</strong></td>
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<td>( \frac{D P_\beta}{D \tau} = \frac{q}{2m} B^{\alpha\beta} S_\alpha )</td>
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The Gravitational analogue of Maxwell’s Equations

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\[
E_{\alpha} = 4\pi \rho c
\]
\[
B_{\alpha} = 0
\]
\[
E_{[\alpha\beta]} = \frac{1}{2} F_{\alpha\beta;\gamma} U^\gamma
\]
\[
B_{[\alpha\beta]} = \frac{1}{2} \star F_{\alpha\beta;\gamma} U^\gamma - 2\pi \epsilon_{\alpha\beta\sigma\gamma} j^\sigma U^\gamma
\]
\[
\mathbb{E}_{\alpha} = 4\pi (2\rho_m + T_{\alpha}^{\alpha})
\]
\[
\mathbb{H}_{\alpha} = 0
\]
\[
\mathbb{E}_{[\alpha\beta]} = 0
\]
\[
\mathbb{H}_{[\alpha\beta]} = -4\pi \epsilon_{\alpha\beta\sigma\gamma} J^\sigma U^\gamma
\]

- Strikingly similar when the setups are stationary in the observer’s rest frame (since \( F_{\alpha\beta;\gamma} U^\gamma \) and \( \star F_{\alpha\beta;\gamma} U^\gamma \) vanish).
The Gravitational analogue of Maxwell’s Equations

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\[ E^\alpha_\alpha = 4\pi \rho_c \]
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\[ E_{[\alpha\beta]} = \frac{1}{2} F_{\alpha\beta;\gamma} U^\gamma \]
\[ B_{[\alpha\beta]} = \frac{1}{2} \ast F_{\alpha\beta;\gamma} U^\gamma - 2\pi \epsilon_{\alpha\beta\sigma\gamma} j^\sigma U^\gamma \]
\[ \mathcal{E}^\alpha_\alpha = 4\pi (2\rho_m + T^\alpha_\alpha) \]
\[ \mathcal{H}^\alpha_\alpha = 0 \]
\[ \mathcal{E}_{[\alpha\beta]} = 0 \]
\[ \mathcal{H}_{[\alpha\beta]} = -4\pi \epsilon_{\alpha\beta\sigma\gamma} J^\sigma U^\gamma \]

► **Charges**: the gravitational analogue of \( \rho_c \) is \( 2\rho_m + T^\alpha_\alpha \)
(\( \rho_m + 3p \) for a perfect fluid) \( \Rightarrow \) in gravity, pressure and all material stresses contribute as sources.
The Gravitational analogue of Maxwell’s Equations

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- **Ampére law**: in stationary (in the observer’s rest frame) setups, equations $B_{[\alpha\beta]}$ and $\mathbb{H}_{[\alpha\beta]}$ match up to a factor of 2 $\Rightarrow$ currents of mass/energy source gravitomagnetism like currents of charge source magnetism.
The Gravitational analogue of Maxwell’s Equations

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E^\alpha_\alpha = 4\pi (2\rho_m + T^\alpha_\alpha) \\
H^\alpha_\alpha = 0 \\
E_{[\alpha\beta]} = 0 \\
H_{[\alpha\beta]} = -4\pi \epsilon_{\alpha\beta\sigma\gamma} J^\sigma U^\gamma
\]

**Absence of electromagnetic-like induction effects in gravity:**

- \( E_{\mu\gamma} \) always symmetric \( \Rightarrow \) no gravitational analogue to Faraday’s law of induction!
The Gravitational analogue of Maxwell’s Equations

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E^\alpha_\alpha = 4\pi (2\rho m + T^\alpha_\alpha) \\
H^\alpha_\alpha = 0 \\
E_{[\alpha\beta]} = 0 \\
H_{[\alpha\beta]} = -4\pi \epsilon_{\alpha\beta\sigma\gamma} j^\sigma U^\gamma
\]

- **Absence of electromagnetic-like induction effects in gravity:**
  - Induction term \( \star F_{\alpha\beta;\gamma} U^\gamma \) in \( B_{[\alpha\beta]} \) has no counterpart in \( H_{[\alpha\beta]} \)
  \( \Rightarrow \) no gravitational analogue to the magnetic fields induced by time varying electric fields.
Magnetic dipole vs Gyroscope

### Electromagnetic Force on a Magnetic Dipole

\[ F^\beta_{EM} = \frac{q}{2m} B^{\alpha\beta} S_\alpha \]

### Gravitational Force on a Spinning Particle

\[ F^\beta_G = -H^{\alpha\beta} S_\alpha \]

The explicit analogy between \( F^\beta_{EM} \) and \( F^\beta_G \) is ideally suited to:

- Compare the two interactions: amounts to compare \( B^{\alpha\beta} \) and \( H^{\alpha\beta} \), which is crystal clear from the equations for tidal tensors:

### Magnetic Tidal Tensor

\[
B_{[\alpha\beta]} = \frac{1}{2} \star F_{\alpha\beta;\gamma} U^\gamma - 2\pi \epsilon_{\alpha\beta\sigma\gamma} j^\sigma U^\gamma
\]

\[ B^\alpha_{\alpha} = 0 \]

### Gravito-Magnetic Tidal Tensor

\[
H_{[\alpha\beta]} = -4\pi \epsilon_{\alpha\beta\sigma\gamma} J^\sigma U^\gamma
\]

\[ H^{\alpha}_{\alpha} = 0 \]
Magnetic dipole vs Gyroscope

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The explicit analogy between $F_{EM}^\beta$ and $F_G^\beta$ is ideally suited to:

- Compare the two interactions: amounts to compare $B_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$, which is crystal clear from the equations for tidal tensors:
  - Unveils similarities between the two forces which allow us visualize, in analogy with the more familiar electromagnetic ones, gravitational effects which are not transparent in the Papapetrou’s original form.
  - and fundamental differences which prove especially enlightening to the understanding of spin-curvature coupling.
Some Fundamental Differences

<table>
<thead>
<tr>
<th>Electromagnetic Force on a Magnetic Dipole</th>
<th>Gravitational Force on a Spinning Particle</th>
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<tr>
<td>$F_{EM}^\beta = \frac{q}{2m} B^{\alpha\beta} S_\alpha$</td>
<td>$F_G^\beta = -H^{\alpha\beta} S_\alpha$</td>
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- $B_{\alpha\beta}$ is linear, whereas $H_{\alpha\beta}$ is not.
- *In vacuum* $H_{[\alpha\beta]} = 0$ (symmetric tensor);
- $B_{[\alpha\beta]} = \frac{1}{2} \star F_{\alpha\beta;\gamma} U^\gamma \neq 0$ (even in vacuum);
- $H_{\alpha\beta} U^\beta = 0$ (spatial tensor) $\Rightarrow F_G^\beta U_\beta = 0$ (it is a spatial force);
- $B_{\alpha\beta} U^\beta \neq 0$ $\Rightarrow F_{EM}^\beta U_\beta \neq 0$ (non-vanishing time projection!)
Symmetries of Tidal tensors

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<td>$B_{[\alpha \beta]} = \frac{1}{2} \star F_{\alpha \beta; \gamma} U^\gamma - 2\pi \epsilon_{\alpha \beta \sigma \gamma} j^\sigma U^\gamma$</td>
<td>$\mathbb{H}<em>{[\alpha \beta]} = -4\pi \epsilon</em>{\alpha \beta \sigma \gamma} J^\sigma U^\gamma$</td>
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- If the fields do not vary along the test particle’s worldline, $\star F_{\alpha \beta; \gamma} U^\gamma = 0$ and the tidal tensors have the same symmetries.
- Allows for a similarity between the two interactions.
Gravitational Spin-Spin Force

\[ F_G^i \simeq \frac{3}{c} \left[ \frac{(\vec{r} \cdot \vec{J})}{r^5} \delta^{ij} + 2 \frac{r^j J_i}{r^5} - 5 \frac{(\vec{r} \cdot \vec{J}) r^j r^i}{r^7} \right] S_j^{\leftrightarrow \mu} F_{EM}^i \]

An analogy already known from linearized theory (Wald, 1972), and usually cast in the form:

\[ \vec{F}_G = -\nabla (\vec{S} \cdot \vec{B}_G) \]
\[ F_G^i \approx \frac{3}{c} \left[ \frac{(\vec{r} \cdot \vec{J})}{r^5} \delta^{ij} + 2 \frac{r^{(ij)} j}{r^5} - 5 \frac{(\vec{r} \cdot \vec{J}) r^i r^j}{r^7} \right] S_j^{\rightarrow \mu} F_{EM}^i \]

An analogy already known from linearized theory (Wald, 1972), and usually cast in the form:

\[ \vec{F}_G = -\nabla (\vec{S} \cdot \vec{B}_G) + \frac{\partial}{\partial t} \left( \epsilon^{ijk} \phi_{,k} \right) S_i \vec{e}_j \]

- Holds only if the gyroscope is at rest and the fields are stationary.
Gravitational Spin-Spin Force

\[ F^i_G \simeq \frac{3}{c} \left[ \frac{\langle \vec{r} \cdot \vec{J} \rangle}{r^5} \delta^{ij} + 2 \frac{r^{(i} J^{j)}}{r^5} - 5 \frac{\langle \vec{r} \cdot \vec{J} \rangle r^i r^j}{r^7} \right] S_j J^{\rightarrow \mu} F^i_{EM} \]

An analogy already known from linearized theory (Wald, 1972), and usually cast in the form:

\[ \vec{F}_G = -\nabla (\vec{S} \cdot \vec{B}_G) + \frac{\partial}{\partial t} \left( \epsilon^{ijk} \phi_{,k} \right) S_i \vec{e}_j \]

▶ Not suitable to describe motion; accounts only for spin-spin coupling.
Gravitational Spin-Spin Force

\[ F_G^i \simeq \frac{3}{c} \left[ \frac{(\vec{r} \cdot \vec{J})}{r^5} \delta^{ij} + 2 \frac{r^i J^j}{r^5} - 5 \frac{(\vec{r} \cdot \vec{J}) r^i r^j}{r^7} \right] S_j \Rightarrow^\mu F_{EM}^i \]

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- Not suitable to describe motion; accounts only for spin-spin coupling.
The gyroscope deviates from geodesic motion even in the absence of rotating sources (e.g. Schwarzschild spacetime).

An effect readily visualized using the explicit analogy (always valid!!):

**Force on a Magnetic Dipole**

\[ F_{EM}^{\beta} = \frac{q}{2m} B^{\alpha\beta} S_{\alpha} \]

**Force on a Gyroscope**

\[ F_{G}^{\beta} = -H^{\alpha\beta} S_{\alpha} \]

- It the magnetic tidal tensor, *as seen by the test particle*, that determines the force exerted upon it;
- Hence the gyroscope deviates from geodesic motion by the same reason that a magnetic dipole suffers a force even in the coulomb field of a point charge: in its “rest” frame, there is a non-vanishing magnetic tidal tensor.
Symmetries of Tidal tensors

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- If the fields vary along the test particle’s wordline, the two interactions differ significantly.
- In vacuum, \( H_{[\alpha\beta]} \) is always symmetric, whereas \( B_{\alpha\beta} \) is not: \( B_{[\alpha\beta]} = \frac{1}{2} \star F_{\alpha\beta;\gamma} U^\gamma \).
Radial Motion in Coulomb Field

The dipole sees a time varying electric field;

Thus, \( B_{[\alpha\beta]} = \frac{1}{2} \ast F_{\alpha\beta}; \gamma U^\gamma \neq 0 \)

\[ F^i_{EM} = \frac{q}{2m} B^i B^{\alpha} \gamma \frac{qQ}{2mr^3} (\vec{v} \times \vec{S})^i \]
Radial Motion in Schwarzschild

- No analogous gravitational effect: $F^\alpha_G = 0 \Rightarrow$ gyroscope moves along a geodesic.
Scalar Invariants

The Riemann tensor (20 independent components) splits irreducibly into three spatial tensors (Louis Bel, 1958):

\[
R_{\alpha\beta}^{\gamma\delta} = 4 \left\{ 2 \tilde{U}_{[\alpha} \tilde{U}_{\gamma]} + g_{[\alpha}^{\gamma} \right\} \tilde{E}_{\beta}]^\delta
+ 2 \left\{ \epsilon_{\alpha\beta\mu\nu} \tilde{H}^{\mu}[\delta \tilde{U}^\gamma] \tilde{U}^\nu + \epsilon^{\gamma\delta\mu\nu} \tilde{H}_{\mu}[\beta \tilde{U}_\alpha] \tilde{U}_\nu \right\}
+ \epsilon_{\alpha\beta\mu\nu} \epsilon^{\gamma\delta\sigma\tau} \tilde{U}^\mu \tilde{U}_\sigma \left\{ \tilde{F}^\nu_{\tau} + \tilde{E}^\nu_{\tau} - g^\nu_{\tau} \tilde{E}^\rho_{\rho} \right\}
\]

\[
\mathbb{F}_{\alpha\beta} \equiv \star R \star_{\alpha\mu\beta\nu} U^\mu U^\nu
\]

- \( \mathbb{E}_{\alpha\beta}, \mathbb{F}_{\alpha\beta} \): spatial, symmetric tensors \( \Rightarrow \) 6 independent components each;
- \( \mathbb{H}_{\alpha\beta} \): spatial, traceless tensor \( \Rightarrow \) 8 independent components
- \( \mathbb{F}_{\alpha\beta} \) has no electromagnetic analogue.
The Riemann tensor (20 independent components) splits irreducibly into three spatial tensors (Louis Bel, 1958):

- in vacuum $F_{\alpha\beta} = -\tilde{E}_{\alpha\beta}$

\[
R_{\alpha\beta}^{\gamma\delta} = 4 \left\{ 2\tilde{U}_{[\alpha} \tilde{U}^{[\gamma} + g_{[\alpha}^{[\gamma} \right\} \tilde{E}_{\beta]}^{\delta]}
+ 2 \left\{ \epsilon_{\alpha\beta\mu\nu} \tilde{H}^{\mu[\delta} \tilde{U}^{\gamma]} \tilde{U}^{\nu} + \epsilon^{\gamma\delta\mu\nu} \tilde{H}_{[\alpha}^{\mu} \tilde{U}_{\beta]} \tilde{U}^{\nu} \right\}
+ \epsilon_{\alpha\beta\mu\nu} \epsilon^{\gamma\delta\sigma\tau} \tilde{U}^{\mu} \tilde{U}^{\nu} \left\{ \tilde{F}^{\nu}_{\tau} + \tilde{E}^{\nu}_{\tau} - g^{\nu}_{\tau} \tilde{E}^{\rho}_{\rho} \right\}
\]

\[
F_{\alpha\beta} \equiv \star R \star_{\alpha\mu\beta\nu} U^{\mu} U^{\nu}
\]

- $E_{\alpha\beta}, F_{\alpha\beta}$: spatial, symmetric tensors $\Rightarrow$ 6 independent components each;
- $H_{\alpha\beta}$: spatial, traceless tensor $\Rightarrow$ 8 independent components
- $F_{\alpha\beta}$ has no electromagnetic analogue.
Scalar Invariants

▷ in vacuum \( \mathbb{F}_{\alpha\beta} = -\mathbb{E}_{\alpha\beta} \)

\[
R^{\gamma\delta}_{\alpha\beta} = 4 \left\{ 2 \tilde{U}_{[\alpha} \tilde{U}^{\gamma]} + g_{[\alpha}^{[\gamma} \tilde{E}^{\delta]}_{\beta] \right\} \\
+ 2 \left\{ \epsilon_{\alpha\beta\mu\nu} \tilde{H}^{\mu[\delta} \tilde{U}^{\gamma]} \tilde{U}^{\nu} + \epsilon^{\gamma\delta\mu\nu} \tilde{H}_{\mu[\beta} \tilde{U}_{\alpha]} \tilde{U}^{\nu} \right\}
\]

▷ \( \mathbb{E}_{\alpha\beta} \) and \( \mathbb{H}_{\alpha\beta} \) completely encode the 14 (6+8) independent components of the Riemann tensor in vacuum (Weyl Tensor).
Scalar Invariants

Although each of these spatial tensors is determined by the 4-velocity $U^\alpha$ of the observer measuring it:

$$E_{\alpha\beta} \equiv R_{\alpha\mu\beta\nu} U^\mu U^\nu$$

$$H_{\alpha\beta} \equiv \star R_{\alpha\mu\beta\nu} U^\mu U^\nu$$

it can be shown that the following expressions are observer independent (in vacuum):

$$E^{\alpha\gamma} E_{\alpha\gamma} - H^{\alpha\gamma} H_{\alpha\gamma} = \frac{1}{8} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$$

$$E^{\alpha\gamma} H_{\alpha\gamma} = \frac{1}{16} R_{\alpha\beta\gamma\delta} \star R^{\alpha\beta\gamma\delta}$$

gravitational tidal tensors form scalar invariants!
Scalar Invariants

▶ in vacuum

\[ E^\alpha \gamma E_{\alpha \gamma} - H^\alpha \gamma H_{\alpha \gamma} = \frac{1}{8} R_{\alpha \beta \gamma \delta} R^{\alpha \beta \gamma \delta} \]
\[ E^\alpha \gamma H_{\alpha \gamma} = \frac{1}{16} R_{\alpha \beta \gamma \delta} \star R^{\alpha \beta \gamma \delta} \]

▶ Formally analogous to the electromagnetic scalar invariants:

\[ \vec{E}^2 - \vec{B}^2 = -\frac{1}{2} F_{\alpha \beta} F^{\alpha \beta} \]
\[ \vec{E} \cdot \vec{B} = -\frac{1}{4} F_{\alpha \beta} \star F^{\alpha \beta} \]
Scalar Invariants

▶ in vacuum

\[
E^\alpha\gamma E_{\alpha\gamma} - H^\alpha\gamma H_{\alpha\gamma} = \frac{1}{8} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}
\]
\[
E^\alpha\gamma H_{\alpha\gamma} = \frac{1}{16} R_{\alpha\beta\gamma\delta} \star R^{\alpha\beta\gamma\delta}
\]

▶ Formally analogous to the electromagnetic scalar invariants:

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\vec{E}^2 - \vec{B}^2 = -\frac{1}{2} F_{\alpha\beta} F^{\alpha\beta}
\]
\[
\vec{E} \cdot \vec{B} = -\frac{1}{4} F_{\alpha\beta} \star F^{\alpha\beta}
\]

▶ This is a purely formal analogy, relating electromagnetic fields with gravitational tidal tensors (which are one order higher in differentiation!)
\[ \vec{E}^2 - \vec{B}^2 = -\frac{1}{2} F_{\alpha\beta} F^{\alpha\beta} \]

\[ \vec{E} \cdot \vec{B} = -\frac{1}{4} F_{\alpha\beta} \star F^{\alpha\beta} \]

- \[ \vec{E} \cdot \vec{B} = 0 \text{ and } \vec{E}^2 - \vec{B}^2 > 0 \Rightarrow \text{there are observers for which the magnetic field } \vec{B} \text{ vanishes.} \]

- \[ \vec{E} \cdot \vec{B} = 0 \text{ and } \vec{E}^2 - \vec{B}^2 < 0 \Rightarrow \text{there are observers for which the electric field } \vec{E} \text{ vanishes.} \]
Scalar Invariants – Electromagnetism

\[
\vec{E}^2 - \vec{B}^2 = -\frac{1}{2} F_{\alpha\beta} F^{\alpha\beta}
\]

\[
\vec{E} \cdot \vec{B} = -\frac{1}{4} F_{\alpha\beta} \star F^{\alpha\beta}
\]

\(\vec{E} \cdot \vec{B} = 0\) and \(\vec{E}^2 - \vec{B}^2 > 0\) \(\Rightarrow\) there are observers for which the magnetic field \(\vec{B}\) vanishes.

\(\vec{E} \cdot \vec{B} = 0\) and \(\vec{E}^2 - \vec{B}^2 < 0\) \(\Rightarrow\) there are observers for which the electric field \(\vec{E}\) vanishes.

\(\vec{E}^2 - \vec{B}^2\) and \(\vec{E} \cdot \vec{B}\) are the only algebraically independent invariants one can define from the Maxwell tensor \(F^{\alpha\beta}\).
In vacuum, one can construct 4 independent scalar invariants from Riemann tensor (would be 14 in general):

\[
E^\alpha\gamma E_{\alpha\gamma} - H^\alpha\gamma H_{\alpha\gamma} = \frac{1}{8} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \equiv \frac{1}{8} R \cdot R
\]

\[
E^\alpha\gamma H_{\alpha\gamma} = \frac{1}{16} R_{\alpha\beta\gamma\delta} \star R^{\alpha\beta\gamma\delta} \equiv \frac{1}{16} R \cdot R^\star
\]

\[
E^\alpha_\beta E^\beta_\gamma E_\alpha^\gamma - 3E^\alpha_\beta H^\beta_\gamma H^\gamma_\alpha = \frac{1}{16} R^{\alpha\beta}_{\lambda\mu} R^{\lambda\mu}_{\rho\sigma} R^{\rho\sigma}_{\alpha\beta} \equiv A
\]

\[
H^\alpha_\beta H^\beta_\gamma H^\gamma_\alpha - 3E^\alpha_\beta E^\beta_\gamma H^\gamma_\alpha = \frac{1}{16} R^{\alpha\beta}_{\lambda\mu} R^{\lambda\mu}_{\rho\sigma} \star R^{\rho\sigma}_{\alpha\beta} \equiv B
\]

- \( R \cdot R^\star = 0 \) and \( R \cdot R \neq 0 \) is not sufficient to ensure that there are observers for which \( H_{\alpha\gamma} \) (or \( E_{\alpha\gamma} \)) vanishes.

- Needs also \( M \equiv l^3 / J^2 - 6 \) to be real positive, where

\[
l \equiv \frac{1}{8} R \cdot R + \frac{i}{8} R \cdot R^\star; \quad J \equiv A - iB
\]
Scalar Invariants – Gravity (Vacuum)

In vacuum, one can construct 4 independent scalar invariants from Riemann tensor (would be 14 in general):

$E^{\alpha\gamma}E_{\alpha\gamma} - H^{\alpha\gamma}H_{\alpha\gamma} = \frac{1}{8} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \equiv \frac{1}{8} R.R$

$E^{\alpha\gamma}H_{\alpha\gamma} = \frac{1}{16} R_{\alpha\beta\gamma\delta} \star R^{\alpha\beta\gamma\delta} \equiv \frac{1}{16} R.\star R$

$E^\alpha E^\beta E^\gamma - 3E^\alpha H^\beta H^\gamma = \frac{1}{16} R^{\alpha\beta\lambda\mu} R^{\lambda\mu\rho\sigma} R^{\rho\sigma \alpha\beta} \equiv A$

$H^\alpha H^\beta H^\gamma - 3E^\alpha E^\beta H^\gamma = \frac{1}{16} R^{\alpha\beta\lambda\mu} R^{\lambda\mu\rho\sigma} \star R^{\rho\sigma \alpha\beta} \equiv B$

- $M \geq 0$ (real), $R.\star R = 0$ and $R.R > 0 \Rightarrow$ there are observers for which $H_{\alpha\gamma}$ vanishes (“Purely Electric” spacetime).

- $M \geq 0$ (real), $R.\star R = 0$ and $R.R < 0 \Rightarrow$ there would be observers for which $E_{\alpha\gamma}$ vanishes (but there are no known “Purely Magnetic” vacuum solutions; conjecture: do not exist).
Kerr metric:
\[
\begin{align*}
E^{\alpha\gamma}E_{\alpha\gamma} - H^{\alpha\gamma}H_{\alpha\gamma} &\approx \frac{6m^2}{r^6} > 0; & I^3 = 6J^2 \text{ (Petrov D)} \\
E^{\alpha\gamma}H_{\alpha\gamma} &\approx \frac{18Jm\cos\theta}{r^7} = 0 \text{ in the plane } \theta = \pi/2
\end{align*}
\]

- In the equatorial plane \( \theta = \pi/2 \), there are observers for which locally \( H_{\alpha\gamma} = 0 \):

\[
\vec{v} = \frac{a}{a^2 + r^2} \vec{e}_\phi;
\]

\[
a \equiv \frac{J}{m}
\]

Mass \( m \)

Angular Momentum \( J \)

- Observers such that \( v^\phi \equiv \frac{U^\phi}{U^t} = \frac{a^2}{a^2 + r^2} \) (not the “co-rotating” observers!)
Impossible in the electromagnetic analog.

A moving dipole sees a time-varying electromagnetic field; thus $B_{\alpha\beta}$ must be non-vanishing (consequence of $\nabla \times \vec{B} = \partial \vec{E}/\partial t$):

$$B_{[\alpha\beta]} = \frac{1}{2} \star F_{\alpha\beta;\gamma} U^\gamma \neq 0 \Rightarrow B_{\alpha\beta} \neq 0$$
Spinning Charge:

\[
\begin{align*}
\vec{E}^2 - \vec{B}^2 &= \frac{q^2}{r^4} - \frac{\mu^2 (5 + 3 \cos 2\theta)}{2r^6} > 0 \\
\vec{E} \cdot \vec{B} &= \frac{2\mu q \cos \theta}{r^5} = 0 \text{ in the plane } \theta = \pi/2
\end{align*}
\]

- In the equatorial plane \( \theta = \pi/2 \), there are observers for which locally \( \vec{B} = 0 \):

- Observers such that \( v^\phi \equiv \frac{U^\phi}{U^t} = \frac{J}{2mr^2} \) (not the “co-rotating” observers!)
For $r \to \infty$, the two velocities asymptotically match! (up to a factor of 2)
▶ No precession:
\[ \vec{B} = 0 \Rightarrow \frac{d\vec{S}}{dt} = 0 \]

▶ There is a Force applied:
\[ B_{\alpha\beta} \neq 0 \Rightarrow F_{EM}^\alpha = \frac{q}{2m} B^{\beta\alpha} S_\beta \neq 0 \]
(consequence of \( \nabla \times \vec{B} = \partial \vec{E} / \partial t \))

▶ Gyroscope precesses:
\[ \frac{d\vec{S}}{dt} \neq 0 \]

▶ No Force on Gyroscope:
\[ H_{\alpha\beta} = 0 \Rightarrow F_G^\alpha = -H^{\beta\alpha} S_\beta = 0 \]
Time projection of $F_{EM}^\alpha$ in the dipole’s proper frame:

$$F_{EM}^\alpha U_\alpha = -B^{i0} \mu_i = \frac{\partial \vec{B}}{\partial t} \cdot \vec{\mu}$$

- The magnetic dipole may be thought as a small current loop.

\[ A = 4\pi a^2; \; I \equiv \text{current through the loop, } \vec{n} \equiv \text{unit vector normal to the loop} \]

- The magnetic dipole moment is given by $\vec{\mu} = IA\vec{n}$
Time projection of $F_{EM}^\alpha$ in the dipole's proper frame:

$$F_{EM}^\alpha U_\alpha = -B^i\mu_i = \frac{\partial \vec{B}}{\partial t} \cdot \vec{\mu} = \frac{\partial \vec{B}}{\partial t} \cdot \vec{n} A l = \frac{\partial \Phi}{\partial t} I$$

- The magnetic dipole may be thought as a small current loop.

\[ \vec{B} \]

\[ \vec{A} \]

\[ \vec{n} \]

(Area of the loop $A = 4\pi a^2$; $I \equiv$ current through the loop, $\vec{n} \equiv$ unit vector normal to the loop)

$$\vec{B} \vec{A} \vec{n} = \Phi \equiv$$ magnetic flux through the loop
Therefore, by Faraday's law of induction:

\[ F_{\text{EM}}^\alpha U_\alpha = -B^{i0} \mu_i = \frac{\partial \vec{B}}{\partial t} \cdot \vec{\mu} = \frac{\partial \vec{B}}{\partial t} \cdot \vec{n}A = \frac{\partial \Phi}{\partial t} I = -I \int_{\text{loop}} \vec{E} \cdot d\vec{s} \]

\[ \vec{E} \equiv \text{Induced electric field} \]

\[ \text{Hence } F_{\text{EM}}^\alpha U_\alpha \text{ is minus the power transferred to the dipole by Faraday's law of induction.} \]
\[ F_{EM}^{\alpha} U_{\alpha} = DE/d\tau \] is minus the power transferred to the dipole by Faraday’s law of induction;

- yields the variation of the energy \( E \) as measured in the dipole’s center of mass rest frame;
- is reflected in a variation of the dipole’s proper mass

\[ m = -P^{\alpha} U_{\alpha} \]
Time Projection of $F_G^\alpha$ — no gravitational induction

Since $H_{\alpha\beta}$ is a spatial tensor, we always have

$$F_G^\alpha U_\alpha = -\frac{dm}{d\tau} = 0$$

- No work is done by induction $\Rightarrow$ the energy of the gyroscope, as measured in its center of mass frame, is constant;
  - the proper mass $m = -P^\alpha U_\alpha$ of the gyroscope is constant.

- Spatial character of gravitational tidal tensors precludes induction effects analogous to the electromagnetic ones.
Time Projection of $F^\alpha_G$ — no gravitational induction

Example: A mass loop subject to the time-varying “gravitomagnetic field” of a moving Kerr Black Hole
Time Projection of $F^\alpha_G$ — no gravitational induction

Example: A mass loop subject to the time-varying “gravitomagnetic field” of a moving Kerr Black Hole

*In the loop’s rest frame:
-No work done on the loop*
Time Projection of $F^\alpha_G$ — no gravitational induction

Example: A mass loop subject to the time-varying “gravitomagnetic field” of a moving Kerr Black Hole

In the loop’s rest frame:
-No work done on the loop
In an arbitrary frame, in which the dipole has 4-velocity $U^\beta = \gamma (1, \vec{v})$, the time component of the force exerted on a magnetic dipole is:

$$(F_{EM})_0 \equiv -\frac{DE}{d\tau} = \frac{F^\beta_{EM} U_\beta}{\gamma} - F^i_{EM} v_i = -\left( \frac{1}{\gamma} \frac{dm}{d\tau} + F^i_{EM} v_i \right)$$

$$\equiv -(P_{mech} + P_{ind})$$

where $E \equiv -P_0$ is the energy of the dipole and we identify:

$\bullet$ $P_{ind} = \frac{1}{\gamma} \frac{dm}{d\tau} = -\frac{F^\beta_{EM} U_\beta}{\gamma} \equiv$ induced power

$\bullet$ $P_{mech} = F^i_{EM} v_i$ “mechanical” power transferred to the dipole by the 3-force $F^i_{EM}$ exerted upon it.
Static Observers — Electromagnetism

\[
(F_{EM})_0 = \frac{q}{2m} B_{\alpha 0} S^\alpha = \star F_{\alpha \gamma;0} U^\gamma S^\alpha = 0
\]

- no work is done on the magnetic dipole.
  - Related to a basic principle from electromagnetism: the total amount of work done by a static magnetic field on an arbitrary system of currents is zero.

\[ \vec{F} = q \vec{v} \times \vec{B} \Rightarrow \vec{F} \perp \vec{v} \]

(Lorentz Force on each individual charge)
Static Observers — Electromagnetism

- When the fields are stationary in the observer’s rest frame, 
  \((F_{EM})_0 = 0 \Rightarrow \) no work is done on the magnetic dipole.
  - \(P_{mech}\) and \(P_{ind}\) exactly cancel out
Static Observers — Electromagnetism

- When the fields are stationary in the observer’s rest frame, 
  \((F_{EM})_0 = 0 \Rightarrow \) no work is done on the magnetic dipole.
  
  - \(P_{mech}\) and \(P_{ind}\) exactly cancel out
Static Observers — Electromagnetism

- When the fields are stationary in the observer’s rest frame, \((F_{EM})_0 = 0 \Rightarrow \) no work is done on the magnetic dipole.
  - \(P_{mech}\) and \(P_{ind}\) exactly cancel out.
In gravity, since those induction effects are absent, such cancellation does not occur:

\[(F_G)_0 = -\frac{DE}{d\tau} = -F^i_G v_i \neq 0\]

Therefore, the stationary observer must measure a non-zero work done on the gyroscope.

- That is to say, a static "gravitomagnetic field" (unlike its electromagnetic counterpart) does work.
- And there is a known consequence of this fact: the spin dependent upper bound for the energy released when two black holes collide, obtained by Hawking (1971) from the area law.
- For the case with spins aligned, from Hawking’s expression one can infer a gravitational spin-spin interaction energy (Wald, 1972).
Static Observers — Gravitational spin interaction

Take the gyroscope to be a small Kerr black hole of spin $S \equiv \sqrt{S^\alpha S_\alpha}$ falling along the symmetry axis of a larger Kerr black hole of mass $m$ and angular momentum $J = am$. 
Take the gyroscope to be a small Kerr black hole of spin $S \equiv \sqrt{S^\alpha S_\alpha}$ falling along the symmetry axis of a larger Kerr black hole of mass $m$ and angular momentum $J = am$. 

Static Observers — Gravitational spin interaction
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Take the gyroscope to be a small Kerr black hole of spin $S \equiv \sqrt{S_\alpha S^\alpha}$ falling along the symmetry axis of a larger Kerr black hole of mass $m$ and angular momentum $J = am$.

The time component of the force acting on the small black hole is given by:

$$(F_G)_0 \equiv \frac{DP_0}{D\tau} = -\frac{dE}{d\tau} = -\frac{2ma(3r^2 - a^2)U^r S}{(r^2 + a^2)^3}$$

Integrating this equation from infinity to the horizon one obtains

$$\int_{\infty}^{r^+} (F_G)_0 \equiv \Delta E = \frac{aS}{2m \left[m + \sqrt{m^2 - a^2}\right]} ,$$

which is precisely Hawking’s spin-spin interaction energy for this particular setup.
Conclusions

- The tidal tensor formalism unveils an exact, fully general analogy between the force on a gyroscope and on a magnetic dipole;
- at the same time it makes transparent both the similarities and key differences between the two interactions;
- The non-geodesic motion of a spinning test particle not only can be easily understood, but also exactly described, by a simple application of this analogy.
- This analogy sheds light on important aspects of spin-curvature coupling;
  - the fact that the mass of a gyroscope is constant (as signaling the absence of gravitational effects analogous to electromagnetic induction);
  - namely, Hawking’s spin dependent upper bound for the energy released on black hole collision (as arising from the fact that gravitomagnetic fields do work);
- Issues concerning previous approaches in the literature were clarified — namely, the limit of validity of the usual linear gravito-electromagnetic analogy, and the physical interpretation of the magnetic parts of the Riemann/Weyl tensors.
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Thank you