Similarities between the Euler problem and the Kerr black hole.

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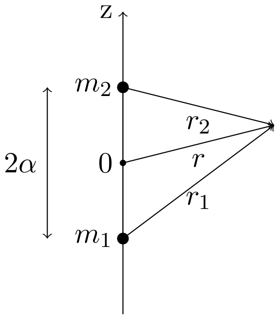
The Newtonian Analogue

Euler potential.
The gravitational field of 2 static masses m_1 and m_2 at a fixed distance 2α apart is:



$$V_E(r) = -\frac{Gm_1}{|\boldsymbol{r} - \alpha\hat{\boldsymbol{z}}|} - \frac{Gm_2}{|\boldsymbol{r} + \alpha\hat{\boldsymbol{z}}|},$$

$$r_1 = |\boldsymbol{r} - \alpha\hat{\boldsymbol{z}}| \text{ and } r_2 = |\boldsymbol{r} + \alpha\hat{\boldsymbol{z}}|.$$

In geometrized units α (which normally has units of length) has spin-parameter dimensions as well.

The Newtonian Analogue

- We will study this problem in Spheroidal coordinates (ξ, η, ϕ) where:

$$\xi = \frac{r_1 + r_2}{2\alpha}, \qquad \eta = \frac{r_2 - r_1}{2\alpha}.$$

The relationship to Cartesian coordinates is:

$$x = \alpha\sqrt{(1+\xi^2)(1-\eta^2)}\cos\phi$$
$$y = \alpha\sqrt{(1+\xi^2)(1-\eta^2)}\sin\phi$$
$$z = \alpha\xi\eta.$$

- In these coordinates the Euler potential becomes:

$$V(\xi, \eta) = -\frac{GM\xi}{\alpha(\xi^2 + \eta^2)}$$

- Euler problem has no horizon.
- The parameter α has different physical significance in these two problems. For the Kerr metric α is the spin of the black hole. In Euler the α parameter is simply a distance. However α has similar implications for the gravitational field of the two problems.
- In Kerr spacetime the prograde and retrograde orbits are distinct due to the frame dragging effect caused by the spin of Kerr metric. In Euler the prograde and retrograde orbits become identical under the transformation $\phi \to -\phi$.

- The Carter constant for the Kerr metric is:

$$Q = p_\theta^2 + \cos^2\theta \left[(1 - E^2)\alpha^2 + \frac{L_z^2}{\sin^2\theta} \right]$$

- In oblate Euler the 3rd Integral of motion (Carter-like constant) yields the same form:

$$Q_N = p_\theta^2 + \cos^2\theta \left[(1 - E^2)\alpha^2 + \frac{L_z^2}{\sin^2\theta} \right]$$

if one makes the following replacements: $\eta \to \cos\theta$, $p_\theta \to \frac{p_\eta}{-\sin\theta}$ and $E_N \to \frac{1-E^2}{2}$.
Both are bilinear with respect to momenta.

Euler (oblate version) has an ISCO radius (like Kerr). If one considers a circular orbit on the equatorial plane, the coresponding effective potential is:

$$V_{eff} = V_E(\rho, z = 0) + \frac{L_z^2}{2\rho^2} = -\frac{GM}{\sqrt{\rho^2 - \alpha^2}} + \frac{L_z^2}{2\rho^2}$$

In spheroidal coordinates:

$$V_{eff} = -\frac{GM}{\alpha\xi} + \frac{L_z^2}{2\alpha^2(\xi^2 + 1)}$$

There is an ISCO when: $V'_{eff} = V''_{eff} = 0$. This leads to:

$$\xi_{\text{ISCO}} = \sqrt{3}$$

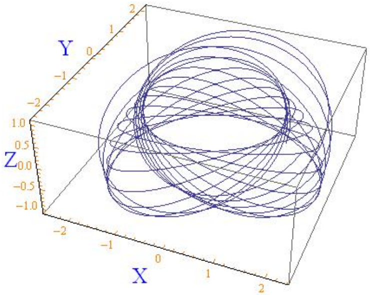
Similarities and Differencies

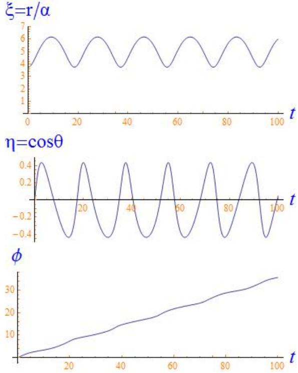
Orbits

Bound geodesic orbits in Euler.

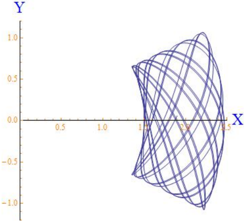
3 fundamental frequencies for Euler orbits:

- Ω_r for radial oscillations (eccentric orbits)
- Ω_θ for oscillations around equator (non-planar orbits)
- Ω_ϕ rotations (orbital winding)





A bound orbit in Euler lays between two ellipses (ξ_{min} ,ξ_{max}) and within a hyperbola ($-\eta_{max}, \eta_{max}$).



$\xi = const$ is an ellipse and $\eta = const$ is an hyperbola. If we view the orbit in spherical coordinates : $r = a\xi$, $\theta = \cos^{-1} \eta$ (the analogue of BL Kerr coordinates) the orbit lays within two spheres and two planes.

Analytical expressions for Euler/Kerr:

$$\Omega_r = \frac{\pi K(k)}{\alpha^2 z_+[K(k)-E(k)]X + YK(k)}$$

$$\Omega_\theta = \frac{\pi\beta\sqrt{z_+}X/2}{\alpha^2 z_+[K(k)-E(k)]X + YK(k)}$$

$$\Omega_\phi = \frac{ZK(k) + XL_z[\Pi(\frac{\pi}{2}, z_-, k) - K(k)]}{\alpha^2 z_+[K(k)-E(k)]X + YK(k)}$$

with K, E, Π elliptic integrals of 1st, 2nd, and 3rd type.

X, Y, Z for Euler

$$X = \int_{r_1}^{r_2} \frac{dr}{\sqrt{V_r}}, \quad Y = \int_{r_1}^{r_2} \frac{r^2}{\sqrt{V_r}} dr, \quad Z = \int_{r_1}^{r_2} \frac{L_z r^2}{(r^2+\alpha^2)\sqrt{V_r}} dr$$

with
$V_r = (E^2-1)r^4 + 2Mr^3 + ((E^2-1)\alpha^2 - Q - L_z^2)r^2 + 2M\alpha^2 r - Q\alpha^2.$

X, Y, Z for Kerr

$$X = \int_{r_1}^{r_2} \frac{dr}{\sqrt{V_r}}, \quad Y = \int_{r_1}^{r_2} \frac{r^2}{\sqrt{V_r}} dr, \quad Z = \int_{r_1}^{r_2} \frac{L_z r^2 - 2Mr(L_z - \alpha E)}{(r^2 - 2Mr + \alpha^2)\sqrt{V_r}} dr$$

with $V_r = (E^2-1)r^4 + 2Mr^3 + \left[(E^2-1)\alpha^2 - Q - L_z^2\right] r^2 + 2M\left[(L_z - \alpha E)^2 + Q\right] r - Q\alpha^2.$

The rest quantities:

- $\beta = a^2(1 - E^2)$,
- while $z_{\pm}$ are related with the θ-oscillation. By setting $z = \cos\theta$, $z_{\pm}$ are the roots of the potential V_θ governing the θ-oscillations.

$$V_\theta = Q - z\left[\alpha^2(1 - E^2) + \frac{L_z^2}{1 - z}\right]$$

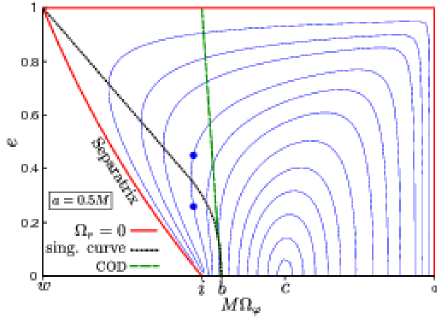
and $k = \sqrt{z_-/z_+}$.

Similarities and Differencies

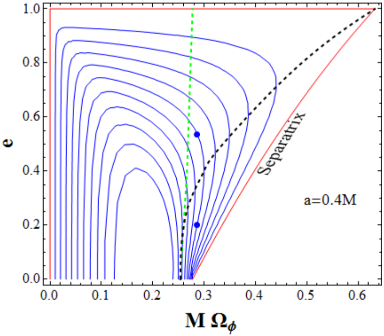
Isofrequency pairs

There are pairs of orbits with the same frequencies triplets.
For equatorial orbits:

Kerr

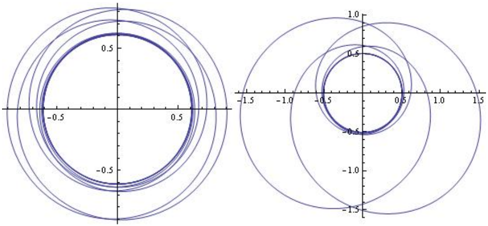


Euler



For Kerr Warburton, N; Barack, L; Sago, N *PRD* **86** 104041

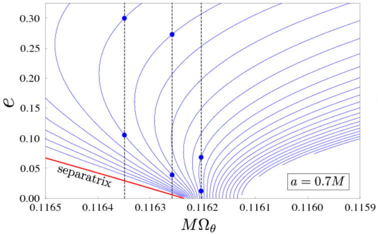
$a = 0.4M, \quad (\Omega_r, \Omega_\phi) = (0.05, 0.286742)$
$(e_1, p_1, \theta_{min1}) = (0.2, 0.7333, \pi/2),$
$(e_2, p_2, \theta_{min2}) = (0.54, 0.0.778, \pi/2).$

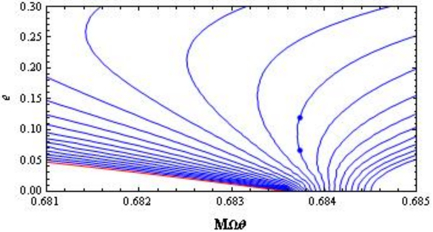


For non equatorial orbits:

Kerr

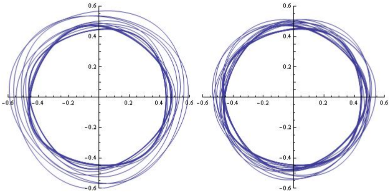
Euler

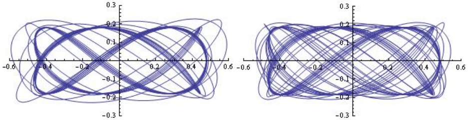




For Kerr Warburton, N; Barack, L; Sago, N *PRD* **86** 104041

$$a = 0.4M, \quad (\Omega_r, \Omega_\theta, \Omega_\phi) = (0.032, 0.683734, 0.4),$$
$$(e_1, p_1, \theta_{min1}) = (0.12, 0.5335, 0.3861),$$
$$(e_2, p_2, \theta_{min2}) = (0.0658, 0.5107, 0.3964).$$



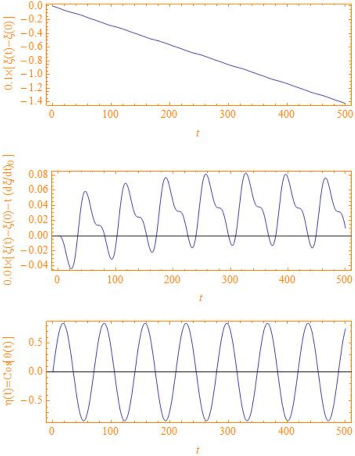


In order to check the argument of Kennefick, D; Ori, A *PRD* **53** 4319 (1996), we apply an atificial force to a test body moving on circular orbit.

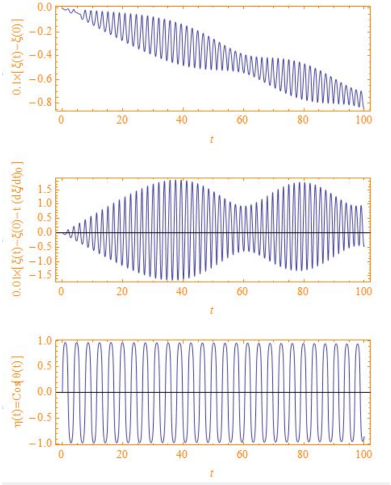
$$\boldsymbol{F} = -k\boldsymbol{v}\frac{1-\eta^2}{\alpha\xi}$$

which obeys the basic symmetries of dissipative back reaction self force: reflection symmetric, ϕ-independent, decrease at large distance, and where $k << 1$ and v is the velocity in spherial coordinates.

$p = 5$, $e = 0$ and $\theta_{min} = 1$



$p = 0.4201357$, $e = 0$ and $\theta_{min} = 1.36$ Resonance $\Omega_r = 2\Omega_\theta$.



Ongoing work

- Using SF in the Newtonian problem to understand the evolution of an orbit near a resonance ($\Omega_r/\Omega_\theta = p/q$).
- If GR is not correct?

THANK YOU