Perturbation and stability of higher dimensional black holes

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GR 100 years in Lisbon
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Introduction

Perturbation analysis:

• **GW emission** from a particle plunging into or orbiting around a BH

• **Stability** problem
  
  Stable ➔ final state of gravitational collapse
  
  Unstable ➔ New branch of solutions

• Information about the geometry: **Quasi-Normal Modes**

• Insights into **Uniqueness/non-uniqueness**

• Attempt to find **new, approximate solutions**
  
  (by deforming an existing solution)
Purpose of this talk

A brief overview of linear perturbation theory of higher dimensional black holes
Two major issues when formulating perturbation theory

- Fixing gauge ambiguity
  - Imposing suitable gauge conditions
  - or
  - Constructing manifestly gauge-invariant variables
Two major issues when formulating perturbation theory

- Fixing gauge ambiguity
  - Imposing suitable gauge conditions
  - Constructing manifestly gauge-invariant variables

- Reduction of perturbation equations to a simple, tractable form \textit{(master equation)}
  - Classifying perturbations into mutually decoupled groups
  - Separating variables
4D master equations

**Static** asymptotically flat vacuum case  
Regge-Wheeler 57  
Zerilli 70  
charge case  
Moncrief  
-- Stability  
Regge-Wheeler 57, Veshveshwara 70 ...

asymptotically AdS/dS case  
Cardoso-Lemos  
--- set of decoupled *self-adjoint* ODEs

**Stationary Rotating** vacuum (Kerr) case  
Teukolsky 72  
--- Stability  
Press-Teukolsky 73  
--- Whiting 89 ...

asymptotically AdS/dS case  
Chambers-Moss 94
Classification Problem in Higher Dimensions

- $D>4$ General Relativity
  No uniqueness like $4D$ GR

Many unstable black (rotating) objects

Dynamical uniqueness theorem
Uniqueness holds for “stable” black objects
Master equations for higher dimensional black holes

- **Rotating BH case** ➔ Not separable in general (e.g., Durkee-Godazgar-Reall) still a long way from having a complete perturbation theory

  Progress in some special cases
  
  Cohomogeneity-one (odd-dim.) Myers-Perry BH
  
  \( D \geq 7 \) Kunduri-Lucietti –Reall 07 (Tensor-modes )
  
  \( 5D \) Murata-Soda 08 (Tensor-Vector-Scalar modes)

  Single-spin (cohomogeneity-two) Myers-Perry
  
  \( D \geq 7 \) Kodama-Konoplya-Zhidenko 09

  Kundt spacetimes (e.g. Near-horizon geometry)
  
  Durkee-Reall 11

- **Static BH case** ➔ simpler and tractable:
  
  -- can reduce to a set of decoupled s.a. ODEs
  
  Kodama-AI 03
Background geometry

\[ \mathcal{M}^D = \mathcal{N}^m \times \mathcal{K}^n \]

\[ ds^2 = g_{ab}(y)dy^a dy^b + r^2(y)d\sigma_n^2 \]

\[ g_{ab}(y) : m - \text{dim spacetime metric} \]

\[ d\sigma_n^2 = \gamma_{ij}(z)dz^i dz^j : n - \text{dim Einstein metric} \]

\[ R_{ij} = (n - 1)K\gamma_{ij} \]

\[ K = \pm 1, 0 \]

-- corresponds to horizon-manifold

This metric can describe a fairly generic class of metrics
$m = 1 \quad y^a \to t \quad \text{FLRW universe} \quad ds^2 = -dt^2 + r(t)^2 d\sigma_n^2$
\( m = 1 \quad y^a \rightarrow t \quad \text{FLRW universe} \quad ds^2 = -dt^2 + r(t)^2 d\sigma_n^2 \)

\( m = 2 \quad y^a \rightarrow (t, r) \quad \text{Static (Schwarzschild-type) black hole} \)

\[ ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\sigma_n^2 \]
\( m = 1 \quad y^a \rightarrow t \quad \text{FLRW universe} \quad ds^2 = -dt^2 + r(t)^2 d\sigma_n^2 \)

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\( m \geq 3 \quad y^a \rightarrow (t, r, \mathbf{y}) \quad \text{Black-brane} \quad ds^2 = dy^2 - f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\sigma_n^2 \)
$m = 1 \quad y^a \to t \quad$ FLRW universe \quad $ds^2 = -dt^2 + r(t)^2 d\sigma_n^2$

$m = 2 \quad y^a \to (t, r) \quad$ Static (Schwarzschild-type) black hole

\[
ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\sigma_n^2
\]

$m \geq 3 \quad y^a \to (t, r, y) \quad$ Black-brane

\[
ds^2 = dy^2 - f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\sigma_n^2
\]

$m = 4 \quad y^a \to (t, r, \theta, \phi) \quad$ Myers-Perry black hole (w/ single rotation)

$r \to r \cos \theta$

\[
ds^2 = \langle 4\text{-dim. Kerr type metric} \rangle + r^2 \cos^2 \theta d\sigma_n^2
\]

Kerr-brane

$r \to \text{const.} \quad ds^2 = \text{Kerr-metric} + d\sigma_n^2$
Cosmological perturbation theory

\[ ds^2 = -dt^2 + r(t)^2 d\sigma_n^2 \] : FLRW background metric

\[ r(t) : \text{scale factor} \]

\[ d\sigma_n^2 = \gamma_{ij}(z) dz^i dz^j \] : homogeneous isotropic time-slice \( n = 3 \)

Perturbations \( \delta g_{\mu\nu}, \delta T_{\mu\nu} \) are decomposed into 3 types according to its tensorial behaviour on time-slice \( (\mathcal{K}^n, \gamma_{ij}) \)

- **Tensor-type**: transverse-traceless (possible only when \( n \geq 3 \))
  \[ \Rightarrow \text{Gravitational Waves} \]
- **Vector-type**: div-free vector \( \Rightarrow \) couple to matter
  e.g. velocity perturbations
- **Scalar-type**: scalar \( \Rightarrow \) couple to matter
  e.g. density perturbations

Gauge-invariant formulation Bardeen 80 Kodama-Sasaki 84
Brane-world cosmology

- AdS - (Black Hole)-Bulk spacetime

\[ ds_{2+n}^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\sigma_n^2 \]

- Brane-world

\[ f(r)\dot{t}^2 - \frac{1}{f(r)}\dot{r}^2 = 1 \]

\[ ds_{1+n}^2 = -d\tau^2 + r^2(\tau)d\sigma_n^2 \]
Brane-world cosmology

- AdS - (Black Hole)-Bulk spacetime

\[ ds^2_{2+n} = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\sigma^2_n \]

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Bulk perturbations induce brane-world cosmological perturbations
--- need to develop a formula for AdS-Black Hole perturbations
--- convenient to decompose bulk perturbations into Tensor-, Vector-, Scalar-type

\[ d\sigma^2_n = \gamma_{ij}(z)dz^idz^j \]

Kodama – Al – Seto ‘00
Black hole background geometry

Static solutions of Einstein-Maxwell + cosmological constant in $D = 2 + n$

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\sigma_n^2$$

$$f(r) = K - \frac{2M}{r^{n-1}} + \frac{Q^2}{r^{2(n-1)}} - \lambda r^2$$

$K = \pm 1, 0$

$M$ ADM-mass

$Q$ charge

$\lambda \propto \Lambda$ Cosmological constant
Basic strategy to derive master equations

(1) Mode-decompose $\delta g_{\mu \nu}$ as

- **Tensor-type**: new component in $D > 4$ case
- **Vector-type**: axial - mode in $D = 4$ case
- **Scalar-type**: polar - mode in $D = 4$ case

(2) Expand $\delta g_{\mu \nu}$ by tensor harmonics $T_{ij}$, $V_i$, $S$ defined on $\mathcal{K}^n$

(3) Write the Einstein equations in terms of the expansion coefficients in 2-dim. spacetime $\mathcal{N}^2$ spanned by $y^a = (t, r)$
Tensor-type perturbations

\[ \delta g_{\mu \nu} = \begin{pmatrix} 0 & 0 \\ 0 & r^{(4-n)/2} \Phi(t,r) \ T_{ij} \end{pmatrix} \]

\[ y^\alpha = (t,r) \]

\[ z^i \]

- \( T_{ij} \): Transverse-Traceless harmonic tensor on \( \mathcal{K}^n \)

\[ (\hat{\Delta}_n + k_T^2)T_{ij} = 0 \quad T^i_i = 0, \quad \hat{D}_j T^j_i = 0 \]

- \( \Phi(t,r) \) is a gauge-invariant variable
Tensor-type perturbations

\[ \delta g_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & r^{(4-n)/2} \Phi(t, r) \mathbb{T}_{ij} \end{pmatrix} \begin{cases} y^a = (t, r) \\ z^i \end{cases} \]

- **\( \mathbb{T}_{ij} \):** Transverse-Traceless harmonic tensor on \( \mathcal{K}^n \)

\[ (\hat{\triangle}_n + k_T^2) \mathbb{T}_{ij} = 0 \quad \mathbb{T}^i_j = 0, \quad \hat{D}_j \mathbb{T}^j_i = 0 \]

- **\( \Phi(t, r) \)** is a gauge-invariant variable

- Einstein’s equations reduce to Master equation \( \mathcal{N}^2 \)

\[
\left( \Box - \frac{V_T}{f} \right) \Phi = 0
\]

\[ V_T \equiv \frac{f}{r^2} \left[ \frac{n(n+2)}{4} f + \frac{n(n+1)M}{r^{n-1}} + k_T^2 - (n - 2)K \right] \]
Vector-type perturbations

\[ \delta g_{\mu\nu} = \begin{pmatrix} 0 & h_a(t, r) \mathbb{V}_i \\ \ast & H(t, r) D_{(i} \mathbb{V}_{j)} \end{pmatrix} \]

\[ y^a = (t, r) \]

\[ z^i \]

- \( \mathbb{V}_i \): Div.-free vector harmonics on \( \mathcal{K}^n \):

\[ (\Delta_n + k_i^2) \mathbb{V}_i = 0, \quad \hat{D}_i \mathbb{V}^i = 0 \]

- Gauge-invariant variable:

\[ F^a := r^{n-2} h^a - \frac{r^n}{2} D^a \left( \frac{H}{r^2} \right) \]

- Einstein’s equations reduce to

\[ \{ \begin{array}{c} D_a F^a = 0 \\
\Box F^a + \cdots = 0 \end{array} \]
Vector-type perturbations

\[ \delta g_{\mu\nu} = \begin{pmatrix} 0 & h_\alpha(t, r) \nabla_i \\ * & H(t, r) \, D(i \nabla_j) \end{pmatrix} \begin{cases} y^\alpha = (t, r) \\ z^i \end{cases} \]

- \nabla_i: Div.-free vector harmonics on \( K^n \): \((\triangle_n + k_V^2) \nabla_i = 0\), \( \hat{D}_i \nabla^i = 0 \)

- Gauge-invariant variable: \( F^\alpha := r^{n-2} h^\alpha - \frac{r^n}{2} D^\alpha \left( \frac{H}{r^2} \right) \)

- Einstein’s equations reduce to

\[
\begin{align*}
D_\alpha F^\alpha &= 0 \quad \cdots \quad (1) \\
\square F^\alpha + \cdots &= 0 \quad \cdots \quad (2)
\end{align*}
\]

(1) There exists \( \Phi(t, r) \) such that \( F^\alpha = \epsilon^{ab} D_b \left( r^{n/2} \Phi \right) \)

(2) Einstein’s equation reduces to Master equation

\[
\left( \square - \frac{V_V}{f} \right) \Phi = 0 \quad V_V = \frac{f}{r^2} \left[ k_V^2 - (n - 1)K + \frac{n(n + 2)}{4} f - \frac{n}{2} r \frac{df}{dr} \right]
\]

-- corresponds to the Regge-Wheeler equation in 4D
Scalar-type perturbations

• Expand $\delta g_{\mu\nu}$ by scalar harmonics $\mathbb{S}$ on $\mathcal{K}^n$: $(\hat{\Delta}_n + k_S^2)\mathbb{S} = 0$

• Construct gauge-invariant variables: $X, Y, Z$ on $\mathcal{N}^2$

• After Fourier transf. wrt ‘$t$’ Einstein’s equations reduce to
  
  \[\left\{\begin{array}{c}
  \text{Set of 1st–order ODEs for } X, Y, Z \\
  \text{A linear algebraic relation among them}
  \end{array}\right.\]
Scalar-type perturbations

- Expand $\delta g_{\mu \nu}$ by scalar harmonics $\mathcal{S}$ on $\mathcal{K}^n$: $(\Delta_n + k_S^2)\mathcal{S} = 0$

- Construct gauge-invariant variables: $X, Y, Z$ on $\mathcal{N}^2$

- After Fourier transf. wrt ‘$t$’ Einstein’s equations reduce to
  
  - Set of 1st order ODEs for $X, Y, Z$
  - A linear algebraic relation among them

  --- such a system can be reduced to a single wave equation

- For a certain linear combination $\Phi(t,r)$ of $X, Y, Z$

  Einstein’s equations reduce to:

  $$(\Box - \frac{V_S}{f})\Phi = 0$$

  -- corresponds to the Zerilli equation in 4D
Stability analysis

• Master equation takes the form:

\[
\frac{\partial^2}{\partial t^2} \Phi = -A \Phi
\]

\[
A = -\frac{d^2}{dr^2} + V
\]
Stability analysis

• Master equation takes the form:

\[
\frac{\partial^2}{\partial t^2} \Phi = -A \Phi
\]

If "A" is a positive self-adjoint operator, the master equation does not admit "unstable" solutions.

\[
A = -\frac{d^2}{dr_*^2} + V
\]

\[
\Phi \propto \exp(-i\omega t)
\]

\[
\omega^2 \int dr_* |\Phi|^2 = \int dr_* \Phi^* A \Phi
\]

--- The black hole is stable
Stability wrt Tensor-type

\[ V_T = \frac{f}{r^2} \left[ \frac{n(n+2)}{4} f + \frac{n(n+1)M}{r^{n-1}} + k_T^2 - (n-2)K \right] > 0 \]

\[ A = -\frac{d^2}{dr^2} + V > 0 \]

Stable
Stability wrt Scalar-type

The potential is \textbf{NOT} positive definite in $D > 4$

- Not obvious to see whether $A = -\frac{d^2}{dr^2} + V$ is positive or not...
Stability proof

- Define \( D := \frac{d}{dr_*} + S \) with some function \( S(r) \)

\[
(\Phi, A\Phi) = -\Phi^* D\Phi \bigg|_{\text{bndry}} + \int dr_* |D\Phi|^2 + \tilde{V} |\Phi|^2
\]

where \( \tilde{V} := V + \frac{dS}{dr_*} - S^2 \)

Boundary terms vanish under the Dirichlet conditions \( \Phi = 0 \)
Stability proof

• Define $D := \frac{d}{dr_*} + S$ w. some function $S(r)$

$$(\Phi, A\Phi) = -\Phi^* D\Phi|_{\text{bndry}} + \int dr_* |D\Phi|^2 + \tilde{V} |\Phi|^2$$

where $\tilde{V} := V + \frac{dS}{dr_*} - S^2$

Boundary terms vanish under the Dirichlet conditions $\Phi = 0$

**Task:** Find $S(r)$ that makes $\tilde{V}$ positive definite

Then, $A$ is uniquely extended to be a *positive* self-adjoint operator
When the horizon manifold $\mathcal{K}^n$ is maximally symmetric

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"OK" $\Rightarrow$ "Stable"

WRT Tensor- and Vector-perturbations $\Rightarrow$ Stable over entire parameter range

WRT Scalar-perturbations $\Rightarrow$ ??? when $Q \neq 0 \land \Lambda \neq 0$
Potential for Scalar-type pert. w. non-vanishing $Q$, $\Lambda$

For extremal and near-extremal case, the potential becomes negative in the immediate vicinity of the horizon.

Numerical study for charged-AdS/dS case Konoplya-Zhidenko 07, 08, 09
Some generalizations and open problems
Static black holes in Lovelock theory

Higher curvature terms involved

\[ L = \sum_{m=0}^{k} c_m \mathcal{L}_m \]

\[ \mathcal{L}_m = \frac{1}{2m} \delta_{\lambda_1\sigma_1} \cdots \delta_{\lambda_m\sigma_m} R^{\lambda_1\sigma_1 \cdots \rho_{m\kappa_m}} R_{\lambda_m\sigma_m \rho_{m\kappa_m}} \]

Equations of motion contain only up to 2\textsuperscript{nd}-order derivatives

\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\sigma_n^2 \]

\[ f(r) = K - X(r)r^2 \]

• Master equations in generic Lovelock theory \(\text{Takahashi – Soda 10}\)

  in Gauss-Bonnet theory \(\text{Dotti – Gleiser 05}\)

• Asymptotically flat, small mass BHs are \textit{unstable} \(\text{wrt}\)

  Tensor-type perturbations (in even-dim.)

  Scalar-type perturbations (in odd-dim.)

• Instability is \textbf{stronger in higher multipoles} rather than low-multipoles

\[ (\Phi, A\Phi) = \int dr_* |D\Phi|^2 + \ell(\ell+n-1) \int dr_* N(r) |\Phi|^2 \]

If \(N(r) < 0\), then \((\Phi, A\Phi) < 0\) for sufficiently large \(\ell\)
Rotating case: Cohomogeneity-2 Myers-Perry BHs

\[ m = 4 \quad ds^2 = \langle 4\text{-dim. Kerr type metric} \rangle + r^2 \cos^2 \theta d\sigma_n^2 \]

symmetry enhance \( U(1)^N \Rightarrow U(1) \times SO(D - 3) \)

Numerical approach to stability analysis

5D bar-mode Shibata-Yoshino 10

--- include the ultra-spinning case

Gregory-Laflamme modes
Axisymmetric perturbation Dias-et. al. 09
Cohomogeneity-2 MP case: Analytic formulation?

\[ \mathcal{N}^4 \quad \mathcal{K}^n \]

\[ m = 4 \quad ds^2 = \langle 4\text{-dim. Kerr type metric} \rangle + r^2 \cos^2 \theta d\sigma_n^2 \]

**Tensor-type perturbations:** A single master scalar variable \( \phi \) on \( \mathcal{N}^4 \) satisfy the same equation for a massless Klein-Gordon field \( n \geq 3 \)

How about **vector-type** and **scalar-type** perturbations?
Cohomogeneity-2 MP case: Analytic formulation?

\[ m = 4 \quad ds^2 = \langle 4\text{-dim. Kerr type metric} \rangle + r^2 \cos^2 \theta d\sigma^2_n \]

Tensor-type perturbations: A single master scalar variable \( \phi \) on \( N^4 \)

satisfy the same equation for a massless Klein-Gordon field

\[ n \geq 3 \]

How about vector-type and scalar-type perturbations?

**Kerr-brane:** 4-dim. Kerr-metric + Ricci flat space

KK-reduction along the Ricci flat space \( K^n \)

⇒ Equations for massive vector/tensor fields

on \( N^4 : 4\text{-dim. Kerr metric} \)

Pani, Gualtieri, Cardoso, Al 15
c.f. Cohomogeneity-1 Myers-Perry BHs

\[ D = \text{odd}, \ J_1 = J_2 = \cdots J_{(D-1)/2} \]

enhanced symmetry: \( \mathbb{R} \times U((D - 1)/2) \)

Perturbation equations reduce to ODEs

Kunduri-Lucietti –Reall 07, Murata-Soda 08
Canonical energy method for initial data

Hollands-Wald 13

Symplectic current

\[ w^a = \frac{1}{16\pi} P_{abcdef} (\gamma_2^a \nabla_d \gamma_1^e - \gamma_1^a \nabla_d \gamma_2^e) \]

Symplectic form

\[ W(\Sigma; \gamma_1, \gamma_2) \equiv \int_{\Sigma} \ast w(g; \gamma_1, \gamma_2) \]

Canonical energy

\[ E(\Sigma, \gamma) \equiv W(\Sigma; \gamma, \xi_K \gamma) - B(\mathcal{B}, \gamma) - C(\mathcal{C}, \gamma) \]

\[ B(\mathcal{B}, \gamma) = \frac{1}{32\pi} \int_{\mathcal{B}} \gamma^{ab} \delta \sigma_{ab} \]

\[ C(\mathcal{C}, \gamma) = -\frac{1}{32\pi} \int_{\mathcal{C}} \tilde{\gamma}^{ab} \delta \tilde{N}_{ab} \]
Canonical energy method for initial data

Symplectic current

\[ w^a = \frac{1}{16\pi} P^{abcdef} (\gamma_{2bc} \nabla_d \gamma_{1ef} - \gamma_{1bc} \nabla_d \gamma_{2ef}) \]

Symplectic form

\[ W(\Sigma; \gamma_1, \gamma_2) \equiv \int_\Sigma \star w(g; \gamma_1, \gamma_2) \]

Canonical energy

\[ \mathcal{E}(\Sigma, \gamma) \equiv W(\Sigma; \gamma, \mathbf{f}_K \gamma) - B(\mathcal{B}, \gamma) - C(\mathcal{C}, \gamma) \]

1) \( \mathcal{E} \) is gauge invariant

2) \( \mathcal{E} \) is monotonically decreasing for any axi-symmetric perturbation

\[ B(\mathcal{B}, \gamma) = \frac{1}{32\pi} \int_{\mathcal{B}} \gamma^{ab} \delta \sigma_{ab} \]

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\[ C(\mathcal{C}, \gamma) = -\frac{1}{32\pi} \int_\mathcal{C} \bar{\gamma}^{ab} \delta \bar{N}_{ab} \]

This method relates Dynamic and Thermodynamic stability criterion and proves Gubser-Mitra conjecture.
Role of symmetry in Stability problem

• Stability of extremal black holes

Examine perturbations of the near-horizon geometry that respect the symmetry (axisymmetry) of the full BH solution

Conjectured by Durkee - Reall 11

When axi-symmetric perturbations on the NHG violate AdS$_2$ -BF-bound on the NHG, then the original extremal BH is unstable

\[ e^{im_I \phi^I} \quad m_I N^I(x) = 0 \]

... supported by numerical results. Dias et al

Proven by use of Canonical energy method Hollands-Al 14
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Proven by use of Canonical energy method Hollands-Al 14

Another application of Canonical energy method

→ Superradiant instability of rotating AdS black holes

Green-Hollands-Al-Wald 15 VIII BH workshop
Summary

• Static HDBHs: Complete formulation for perturbations
Summary

- Static HDBHs: Complete formulation for perturbations

- Rotating HDBHs:
  -- Still a long way from having a complete formulation
  -- Considerable progress recently made for some special cases
Summary

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• Interplay between
  Exact solutions + Perturbation analysis
  Numerical Analysis
  Mathematical Theorems
Interplay between Exact solutions + Perturbation
Numerical Analysis
Mathematical Theorems

1915  Einstein equations
1915  Schwarzschild Solution
1939  Oppenheimer-Snyder
1957  Regge-Wheeler equation
1963  Kerr solution
1965  Singularity Theorems
1970  Zerilli equation
1973  Teukolsky equation
1975  BH Thermodynamiccs laws
1982  Uniqueness Theorem
1983  Positive Energy Theorem
1985  Accurate method to BH QNMs

Exact Solutions + Perturbation analysis

Mathematica Theorems

Numerical Approach
Exact solutions + Perturbation

1986  Myers-Perry Solution

1993  BTZ Solution
Gregory-Laflamme Instability
Choptuick’s critical collapse in Numerical GR
BSSN system in Numerical GR

1997  AdS-CFT correspondence

1998  Brane-world scenario

2001  Emparan-Reall black ring
HD BH Perturbation theory: This talk
Doubly spinning black ring
Black saturn
Multiple- black rings

2015  Black-lens Kunduri-Lucietti
**Numerical Analysis**

1986  
Myers-Perry Solution

1993  
BTZ Solution
  
  Gregory-Laflamme Instability
  
  Choptuick’s critical collapse in Numerical GR
  
  BSSN system in Numerical GR

1997  
AdS-CFT correspondence

1998  
Brane-world scenario

2001  
Emparan-Reall black ring
  
  High energy collisions of BHs Sperhake et al
  
  Axisymmetric perturbation of MP BH – Dias et. al
  
  Bar-mode instability of MP BH Shibata-Yoshino
  
  Black-String final fate Lehner-Pretorius

2015  
Instability of AdS spacetimes – Bizon-Rostworowsky
Mathematical Theorems

1986  Myers-Perry Solution

1993  BTZ Solution
       Gregory-Laflamme Instability
       Choptuick’s critical collapse in Numerical GR
       BSSN system in Numerical GR

1997  AdS-CFT correspondence

1998  Brane-world scenario

2001  Emparan-Reall black ring
       HD generalization of BH Topology Theorem
       HD generalization of BH rigidity (Symmetry) Theorem
       HD Uniqueness/Non-uniqueness Theorems

2015  

Interplay between Exact solutions + Perturbation
Numerical Analysis
Mathematical Theorems

1986
Myers-Perry Solution

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BTZ Solution
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1997
AdS-CFT correspondence

1998
Brane-world scenario

2001
Emparan-Reall black ring

Higher dimensional General Relativity

2015
At GR Centenary

- **Perturbation theory** has played a major role in understanding basic properties—e.g. stability—of *exact solutions* at hand.

- **Numerical Approach** has become more important to reveal interesting properties of complicated systems and/or to deal with more realistic models.

- **Mathematical theorems** as guide lines.

- Interplay between
  - **Numerical Approach**
  - **Mathematical Theorems** and
  - **Exact solution + Perturbation analysis**
will be getting more and more important.