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Entropy of matter systems in strong gravitational fields and the black hole limit

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Outline

1. Introduction
2. Dynamics of shells: the simpest spacetime after vacuum
3. Thermodynamics of shells: generics
4. Thermodynamics of shells: independent variables and equations of state for the shell's p, T, and Φ
5. Thermodynamics of shells: entropy of thin shells
6. Thermodynamics of shells: examples
7. Thermodynamics of shells: the black hole limit
8. Thermodynamics of extremal shells: the extremal black hole limit
9. References

1. Introduction

- Entropy is related to degrees of freedom. Matter entropy is related to the volume, e.g., Sakur-Tetrode entropy (1912), the entropy of a monatomic classical ideal gas which incorporates quantum considerations $S = N\left(\ln\left[\frac{V}{N}\left(\frac{m}{3\pi\hbar^2}\frac{U}{N}\right)^{3/2}\right]+\frac{5}{2}\right)$.
- Black hole entropy is in the area, the Bekenstein-Hawking entropy $S = \frac{1}{4}\frac{A_+}{A_p}$, $A_p = \hbar$, the Planck area ($G = 1$, $c = 1$, $k_B = 1$). Points to the ultimate degrees of freedom are in the area not volume. Works of 1970s.
- This is well established for nonextremal black holes: thermodynamics of black holes, Euclidean formulation and path integral approach to statistical mechanics.
- Not so for extremal black holes. The Euclidean formulation shows that $S = 0$ due to trivial topology (Hawking, Horowitz, Ross 1995, Teitelboim 1995). On the other hand string theory formulation of black holes shows $S = \frac{1}{4}\frac{A_+}{A_p}$ (Strominger, Vafa 1996). There is a problem here.
- We use matter to study black hole entropy. Use the simplest form of matter: a shell. Amazingly, it reflects and gives a solution to the debate.

2. Dynamics of shells: the simpest spacetime after vacuum

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}\,, \quad \nabla_\beta F^{\alpha\beta} = 4\pi J^\alpha \qquad\qquad (G=1\,,\ c=1).$$

In the inner region $\mathscr{V}_i$ ($r \le R$) we assume the spacetime is flat, i.e.

$$ds_i^2 = g^i_{\alpha\beta} dx^\alpha dx^\beta = -dt_i^2 + dr^2 + r^2\, d\Omega^2\,.$$

In the outer region $\mathscr{V}_o$ ($r \ge R$), the spacetime is Reissner-Nordström

$$ds_o^2 = g^o_{\alpha\beta} dx^\alpha dx^\beta = -\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) dt_o^2 + \frac{dr^2}{1 - \frac{2m}{r} + \frac{Q^2}{r^2}} + r^2 d\Omega^2\,.$$

On the hypersurface itself, $r = R$, the metric h_{ab} is that of a 2-sphere plus time,

$$ds_\Sigma^2 = h_{ab} dy^a dy^b = -d\tau^2 + R^2(\tau) d\Omega^2\,.$$

The metric h_{ab} is the induced metric,

$$h^i_{ab} = g^i_{\alpha\beta}\, e_i{}^\alpha{}_a\, e_i{}^\beta{}_b\,, \quad h^o_{ab} = g^o_{\alpha\beta}\, e_o{}^\alpha{}_a\, e_o{}^\beta{}_b\,,$$

where $e_i{}^\alpha{}_a$ and $e_o{}^\alpha{}_a$ are tangent vectors to the hypersurface viewed from the inner and outer regions, respectively.

2. Dynamics of shells: the simpest spacetime after vacuum

$$[h_{ab}] = 0\,,$$

where [ ] means the jump in the quantity across the hypersurface.

$$S^a{}_b = -\frac{1}{8\pi}\left([K^a{}_b] - [K]h^a{}_b\right)\,,$$

$$K^a_{i\,b} = \nabla_\beta\, n^i_\alpha\, e^\alpha_{i\;c}\, e^\beta_{i\;b}\, h^{ca}_i\,, \quad K^a_{o\,b} = \nabla_\beta\, n^o_\alpha\, e^\alpha_{o\;c}\, e^\beta_{o\;b}\, h^{ca}_o\,,$$

where ∇_β is the symbol for covariant derivative and n^i_α n^o_α, are the inner and outer normals to the shell,

Find

$$\sigma = \frac{1 - \sqrt{1 - \frac{2m}{R} + \frac{Q^2}{R^2}}}{4\pi R}\,,$$

$$p = \frac{1 - \frac{m}{R} - \sqrt{1 - \frac{2m}{R} + \frac{Q^2}{R^2}}}{8\pi R}\,.$$

Can envisage as an equation of state $p = p(\sigma, R, Q)$.

2. Dynamics of shells: the simpest spacetime after vacuum

The shell's redshift function k is

$$k = \sqrt{1 - \frac{2m}{R} + \frac{Q^2}{R^2}}\,.$$

Then

$$\sigma = \frac{1-k}{4\pi R}\,, \quad p = \frac{R^2(1-k)^2 - Q^2}{16\pi R^3 k}\,.$$

Define rest mass M as

$$\sigma = \frac{M}{4\pi R^2}\,, \quad \text{so} \quad M = R\,(1-k)\,.$$

One is led to an equation for the ADM mass m,

$$m = M - \frac{M^2}{2R} + \frac{Q^2}{2R}\,.$$

This equation is intuitive in physical grounds as it states that the total energy m of the shell is given by its mass M minus the energy required to built it against the action of gravitational and electrostatic forces, i.e., $-\frac{M^2}{2R} + \frac{Q^2}{2R}$.

2. Dynamics of shells: the simpest spacetime after vacuum

The gravitational radius r_+ and the Cauchy horizon r_- of the shell spacetime are

$$r_+ = m + \sqrt{m^2 - Q^2}\,, \quad r_- = m - \sqrt{m^2 - Q^2}\,.$$

The gravitational radius r_+ is also the horizon radius when the shell radius R is inside r_+, i.e., the spacetime contains a black hole. Inverting

$$m = (r_+ + r_-)\,, \quad Q = \sqrt{r_+ r_-}\,.$$

Then k can be written as

$$k = \sqrt{\left(1 - \frac{r_+}{R}\right)\left(1 - \frac{r_-}{R}\right)}\,.$$

The gravitational area A_+ and the area A of the shell are

$$A_+ = 4\pi\, r_+^2\,, \quad A = 4\pi R^2$$

Shell should obey

$$R \geq r_+.$$

2. Dynamics of shells: the simpest spacetime after vacuum

The Faraday-Maxwell tensor $F_{\alpha\beta}$ is defined in terms of an electromagnetic four-potential A_α by

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \,.$$

To use the thin shell formalism related to the electric part we need to specify the vector potential A_α in each side of the shell. Assume an electric ansatz

$$A_\alpha = (-\phi, 0, 0, 0) \,.$$

In the inner and outer regions get

$$\phi_i = \frac{Q}{R} + \text{constant} \,, \; r \leq R \,, \quad \phi_o = \frac{Q}{r} + \text{constant} \,, \; r \geq R \,.$$

where Q is a constant, to be interpreted as the conserved electric charge. Now

$$A_a = A_\alpha \, e^\alpha_a$$

is the projected 4-potential intrinsic to the shell. Then

$$[A_a] = 0 \,,$$

gives at R

$$\phi_o = \phi_i \,, \quad r = R \,.$$

2. Dynamics of shells: the simpest spacetime after vacuum

The tangential components F_{ab} of the electromagnetic tensor $F_{\alpha\beta}$ must change smoothly across Σ,

$$[F_{ab}] = 0\,,$$

with

$$F^i_{ab} = F^i_{\alpha\beta}\, e_i{}^\alpha{}_a\, e_i{}^\beta{}_b\,, \quad F^o_{ab} = F^o_{\alpha\beta}\, e_o{}^\alpha{}_a\, e_o{}^\beta{}_b\,,$$

while the normal components $F_{a\perp}$ must change by a jump as,

$$[F_{a\perp}] = 4\pi\sigma_e u_a\,,$$

where

$$F^i_{a\perp} = F^i_{\alpha\beta}\, e_i{}^\alpha{}_a\, n_i^\beta\,, \quad F^o_{a\perp} = F^o_{\alpha\beta}\, e_o{}^\alpha{}_a\, n_o^\beta\,,$$

and $\sigma_e u_a$ is the surface electric current, with σ_e being the density of charge and u_a its 3-velocity, defined on the shell. Then on the shell,

$$\frac{\partial\phi_o}{\partial r} - \frac{\partial\phi_i}{\partial r} = -4\pi\sigma_e\,, \quad r = R\,.$$

Finally

$$\frac{Q}{R^2} = 4\pi\sigma_e\,.$$

3. Thermodynamics of shells: generics

Now the shell is hot. The fluid is still perfect. Should then turn to the thermodynamic side and to the calculation of the entropy of the shell. The shell possesses a well defined temperature T and an entropy S which is a function of M, A, Q, i.e.,

$$S = S(M, A, Q)\,.$$

The first law of thermodynamics is then

$$TdS = dM + pdA - \Phi dQ\,.$$

To find S, one needs three equations of state

$$p = p(M, A, Q)\,,$$

$$\beta = \beta(M, A, Q)\,,$$

$$\Phi = \Phi(M, A, Q)\,,$$

where

$$\beta \equiv \frac{1}{T}\,.$$

T and Φ play a role of integration factors, i.e., there will be integrability conditions.

3. Thermodynamics of shells: generics

The integrability conditions must be specified in order to guarantee the existence of an expression for the entropy, so dS is exact. They are

$$
\begin{aligned}
\left(\frac{\partial\beta}{\partial A}\right)_{M,Q} &= \left(\frac{\partial\beta p}{\partial M}\right)_{A,Q}, \\
\left(\frac{\partial\beta}{\partial Q}\right)_{M,A} &= -\left(\frac{\partial\beta\Phi}{\partial M}\right)_{A,Q}, \\
\left(\frac{\partial\beta p}{\partial Q}\right)_{M,A} &= -\left(\frac{\partial\beta\Phi}{\partial A}\right)_{M,Q}.
\end{aligned}
$$

These determine the relations between the three equations of state of the system, here the shell.

From the first law of thermodynamics one can perform a thermodynamic study of the local intrinsic stability of the shell,

$$
\left(\frac{\partial^2 S}{\partial M^2}\right)_{A,Q} \leq 0\,, \quad \left(\frac{\partial^2 S}{\partial M^2}\right)\left(\frac{\partial^2 S}{\partial Q^2}\right) - \left(\frac{\partial^2 S}{\partial M\partial Q}\right)^2 \geq 0\,,
$$

plus four other equations.

4. Thermodynamics of shells: independent variables and equations of state for the shell's p, T, and Φ

From now onwards we work with the three independent variables

$$(M, R, Q)\,.$$

R is simpler then A, ($R = \sqrt{\frac{A}{4\pi}}$).
We should now envisage all quantitites as functions of (M, R, Q),

$$m(M,R,Q) = M - \frac{M^2}{2R} + \frac{Q^2}{2R}\,,$$

$$r_+(M,R,Q) = m(M,R,Q) + \sqrt{m(M,R,Q)^2 - Q^2}\,,$$

$$r_-(M,R,Q) = m(M,R,Q) - \sqrt{m(M,R,Q)^2 - Q^2}\,,$$

$$k(r_+(M,R,Q), r_-(M,R,Q), R) = \sqrt{\left(1 - \frac{r_+(M,R,Q)}{R}\right)\left(1 - \frac{r_-(M,R,Q)}{R}\right)}\,.$$

4. Thermodynamics of shells: independent variables and equations of state for the shell's p, T, and Φ

The pressure equation of state:

Expressing the pressure equation of state as a function of (M,R,Q)

$$p(M,R,Q) = \frac{M^2 - Q^2}{16\pi R^2 (R-M)} \,.$$

Changing from the variables (M,R,Q) to (r_+,r_-,R) which is more useful find

$$p(r_+,r_-,R) = \frac{R^2(1-k)^2 - r_+ r_-}{16\pi R^3\, k} \,,$$

where k can be envisaged as $k = k(r_+,r_-,R)$ and r_+ and r_- are functions of (M,R,Q).

This equation is a pure consequence of the Einstein equation, encoded in the junction conditions.

4. Thermodynamics of shells: independent variables and equations of state for the shell's p, T, and Φ

The temperature equation of state:

Now we have the integrability condition $\left(\frac{\partial\beta}{\partial A}\right)_{M,Q} = \left(\frac{\partial\beta p}{\partial M}\right)_{A,Q}$. Changing from the variables (M,R,Q) to (r_+,r_-,R) it becomes

$$\left(\frac{\partial\beta}{\partial R}\right)_{r_+,r_-} = \beta\,\frac{R(r_+ + r_-) - 2r_+r_-}{2R^3k^2}\,,$$

which has the solution

$$\beta(r_+,r_-,R) = b(r_+,r_-)\,k\,,$$

where k is the redshift function, function of r_+, r_-, and R.

Also $b(r_+,r_-) \equiv \beta(\infty,r_+,r_-)$ is an arbitrary function, representing the inverse of the temperature of the shell if its radius were infinite.

Hence, our formalism recovers Tolman's formula for the temperature of a body in curved spacetime. The arbitrariness of this function is due to the fact that the matter fields of the shell have to be specified. Note that b and k are still functions of (M,R,Q).

4. Thermodynamics of shells: independent variables and equations of state for the shell's p, T, and Φ

The electric potential equation of state:
The integrability conditions give

$$R^2 \left(\frac{\partial \Phi k}{\partial R}\right)_{r_+,r_-} - \sqrt{r_+ r_-} = 0\,,$$

where again k can be envisaged as $k = k(r_+, r_-, R)$. The solution is

$$\Phi(r_+, r_-, R) = \frac{\phi(r_+, r_-) - \frac{\sqrt{r_+ r_-}}{R}}{k}\,,$$

where $\phi(r_+, r_-) \equiv \Phi(\infty, r_+, r_-)$ is an arbitrary function that corresponds to the electric potential of the shell if it were at infinity. Φ is the difference in the electric potential ϕ between infinity and R, blueshifted from infinity to R.
It is convenient to define $c(r_+, r_-) \equiv \frac{\phi(r_+, r_-)}{Q}$ or $c(r_+, r_-) \equiv \frac{\phi(r_+, r_-)}{\sqrt{r_+ r_-}}$. So

$$\Phi(r_+, r_-, R) = \frac{c(r_+, r_-) - \frac{1}{R}}{k} \sqrt{r_+ r_-}\,.$$

5. Thermodynamics of shells: entropy of thin shells

Have all necessary information to calculate the entropy S of the shell. Inserting the equations of state for pressure, temperature, and electric potential, into the first law $TdS = dM + pdA + \Phi dQ$ find

$$dS = b(r_+, r_-)\frac{1 - c(r_+, r_-)r_-}{2}dr_+ + b(r_+, r_-)\frac{1 - c(r_+, r_-)r_+}{2}dr_- \,.$$

It has its own integrability condition if dS is to be an exact differential,

$$\frac{\partial b}{\partial r_-}(1 - r_- c) - \frac{\partial b}{\partial r_+}(1 - r_+ c) = \frac{\partial c}{\partial r_-}br_- - \frac{\partial c}{\partial r_+}br_+ .$$

So

$$S = S(r_+, r_-)\,,$$

the entropy is a function of r_+ and r_-. In fact S is a function of (M, R, Q), but dependence has to be through $r_+(M, R, Q)$ and $r_-(M, R, Q)$,

$$S(M, R, Q) = S(r_+(M, R, Q), r_-(M, R, Q))\,.$$

To obtain a specific expression for S one can choose either b or c, the other function comes from integrability. Since it is a differential equation there is some freedom.

6. Thermodynamics of shells: examples

1. $b(r_+, r_-) = 2a(r_+ + r_-)^\alpha$,
   $c(r_+, r_-) = 2d\,\frac{(r_+ r_-)^\delta}{(r_+ + r_-)^\alpha}$.
   Then,
   $S(r_+, r_-) = a\left[\frac{(r_+ + r_-)^{\alpha+1}}{\alpha+1} - d\,\frac{(r_+ r_-)^{\delta+1}}{\delta+1}\right]$.
2. $b(r_+, r_-) = \frac{h(r_+)}{r_+ - r_-}$,
   $c(r_+, r_-) = \frac{1}{r_+}$.
   Then,
   $S(r_+) = \frac{1}{2}\int_0^{r_+} \frac{h(x)}{x}\,dx$.
3. $b(r_+, r_-) = \frac{h(r_-)}{r_+ - r_-}$,
   $c(r_+, r_-) = \frac{1}{r_-}$.
   Then,
   $S(r_-) = \frac{1}{2}\int_0^{r_-} \frac{h(x)}{x}\,dx$.
4. $b(r_+, r_-) = b_0$,
   $c(r_+, r_-) = c(r_+ r_-)$.
   Then,
   $S(r_+, r_-) = \frac{b_0}{2}\left(r_+ + r_- - \int_0^{r_+ r_-} c(x)\,dx\right)$.

7. Thermodynamics of shells: the black hole limit

Now, the black hole limit is $R \to r_+$. The shell hovering at its own gravitational radius.
The shell adjusts to the environmental spacetime: quantum fields and back-reaction diverge unless choose the black hole Hawking T_{bh} for the shell,

$$b(r_+, r_-) = \frac{1}{T_{\mathrm{bh}}} = \frac{4\pi}{A_p} \frac{r_+^2}{r_+ - r_-} \,.$$

Choose also

$$c(r_+, r_-) = \frac{1}{r_+} \,.$$

Get

$$S = \frac{1}{4} \frac{A_+}{A_p} \,,$$

the Bekenstein-Hawking entropy. The pressure and the thermodynamic electric potential go to infinity as $1/k$. The local inverse temperature goes to zero as k, and the local temperature of the shell goes to infinity as $1/k$. These well controlled infinities cancel out in the first law to give the entropy.
As $A = A_+$ all the shell's fundamental degrees of freedom have been excited.

7. Thermodynamics of shells: the black hole limit

There are similarities between the thin shell approach and the black hole mechanics approach. These are evident if we express the differential of the entropy of the charged shell in terms of the black hole ADM mass m and charge Q, given in terms of the variables (r_+, r_-). The differential for the entropy of the shell reads in these variables

$$T_0 dS = dm - c\, Q\, dQ\,,$$

where we have defined $T_0 \equiv 1/b(r_+, r_-)$ which is the temperature the shell would possess if located at infinity. Here, $T_0 = 1/b(r_+, r_-)$ and $c = c(r_+, r_-)$ should be seen as $T_0(m, Q) = 1/b(m, Q)$ and $c(m, Q)$, respectively, since r_+ and r_- are functions of m and Q. As we have seen, if we take the shell to its gravitational radius, we must fix $T_0 = T_{\rm bh}$ and $c = 1/r_+$. This suggests that Q/r_+ should play the role of the black hole electric potential $\Phi_{\rm bh}$, which in fact is true. So the conservation of energy of the shell is expressed as

$$T_{\rm bh} dS_{\rm bh} = dm - \Phi_{\rm bh}\, dQ\,.$$

We thus see that the first law of thermodynamics for the shell at its own gravitational radius is equal to the energy conservation for the black hole.

8. Thermodynamics of extremal shells: the extremal black hole limit

Here,
$$ds_o^2 = -\left(1-\frac{m}{r}\right)^2 dt_o^2 + \frac{dr^2}{\left(1-\frac{m}{r}\right)^2} + r^2 d\Omega^2\,, \quad r \geq R\,,$$
$$r_+ = r_- = m = Q = M\,. \quad \sigma = \frac{M}{4\pi R^2}\,, \quad p = 0\,.$$

The first law of thermodynamics
$$TdS = dM + pdA - \Phi dQ\,,$$
gives now
$$dS = \beta\,(1-\Phi)\,dr_+\,.$$
Integrability gives
$$\beta\,(1-\Phi) = s(r_+)\,.$$
Thus $S = S(r_+)$ for $R \geq r_+$. In particular in the black hole limit
$$S = S(r_+)\,, \quad R = r_+\,,$$
Can argue
$$0 \leq S(r_+) \leq \frac{1}{4}\frac{A_+}{A_p}\,,$$
or $0 \leq S(r_+) \leq \pi r_+^2$.

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