Newtonian Gravitation, General Relativity, and Quantum Gravity

José Sande Lemos

CENTRA – IST

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1. Constants in physics

- Newtonian gravitation: G
- General relativity: G, c
  (Einstein)
- Quantum mechanics: h
  (Bohr, Schrödinger, Heisenberg)

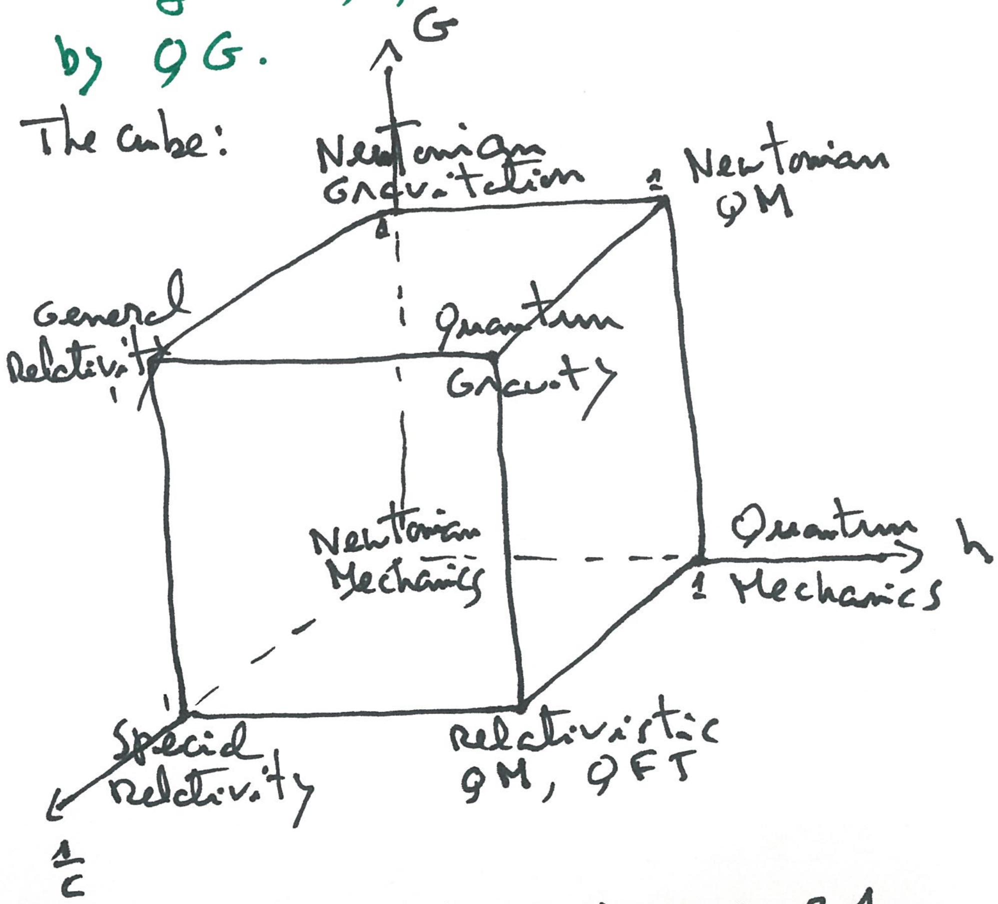
Newtonian grav. only has G, since velocities $v \ll c$. Joining c from special relativity get GR.

Quantum mechanics only has h. Valid only for $v \ll c$. Joining c get relativistic quantum mechanics. (Dirac)

GR has G and c. Not valid for quantum systems, i.e., small. Joining h get quantum gravity, G, c, h.

- Black holes belong to GR but what they hide, spacetime singularities belong to quantum gravity.
- The Universe started with a singularity, surely to be explained by QG.

The cube:



$G = 0$ or 1
$\frac{1}{c} = 0$ or 1

$h = 0$ or 1
in Planck units

Why choose G, c, and h?

- Only gravitation is universal
  G units $L^3 M^{-1} T^{-2}$
- All that moves is described by the principles of special relativity
  c units $L T^{-1}$
- All that exists is described by quantum mechanics
  h units $M L^2 T^{-1}$

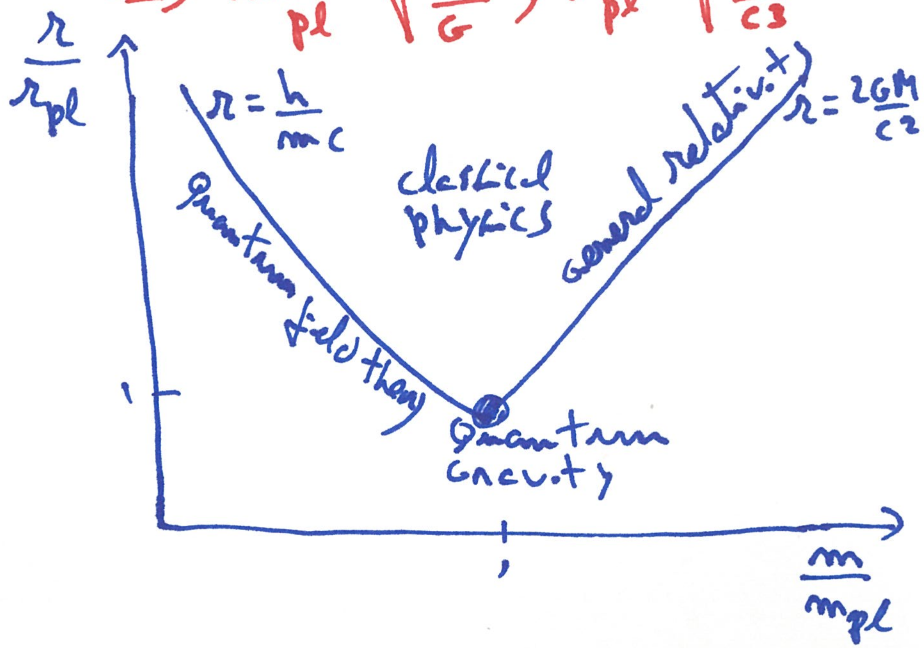
At which scale one finds the most possible simple processes that involve gravity?
The scale is set by the 3 constants G, c, h and is called the Planck scale.

$r_{pl} = \sqrt{\frac{G h}{c^3}}$ $(10^{-33}\,\text{cm})$ $t_{pl} = \sqrt{\frac{G h}{c^5}}$ $(10^{-43}\,\text{s})$ $m_{pl} = \sqrt{\frac{h c}{G}}$ $(10^{-5}\,\text{g})$

Let us see this more deeply.
Two formulas that relate mass and radius in physics:

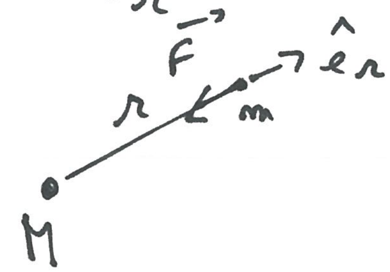
- One is quantum r_{compt}
  $$r_{compt} = \frac{h}{m c}$$
- The other is from GR r_{schw}
  $$r_{schw} = \frac{2 G m}{c^2}$$
- They meet at $\frac{h}{m c} = \frac{G m}{c^2}$

$\Longrightarrow m_{pl} = \sqrt{\frac{h c}{G}}, \quad r_{pl} = \sqrt{\frac{G h}{c^3}}$



2. Advanced Newtonian gravitation

- Newton's force law of gravitation is $\vec{F} = -\frac{GMm}{r^2}\,\hat{e}_r$

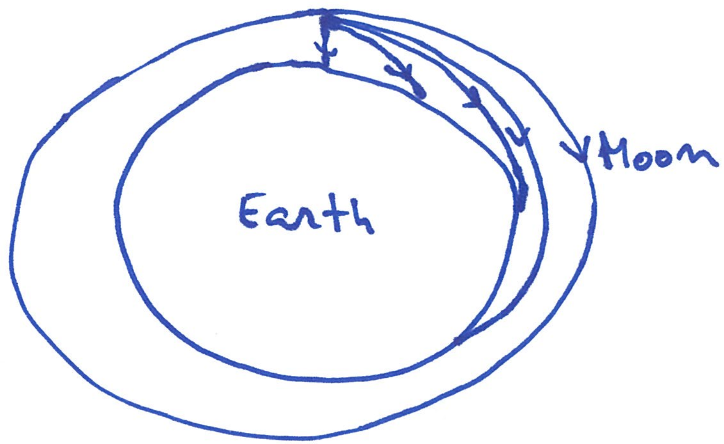


force per unit mass m is $\vec{g} = -\frac{GM}{r^2}\,\hat{e}_r$

It is conservative so find gravitational potential ϕ

$$\phi = -\frac{GM}{r} \qquad \text{with} \qquad \vec{g} = -\vec{\nabla}\phi$$

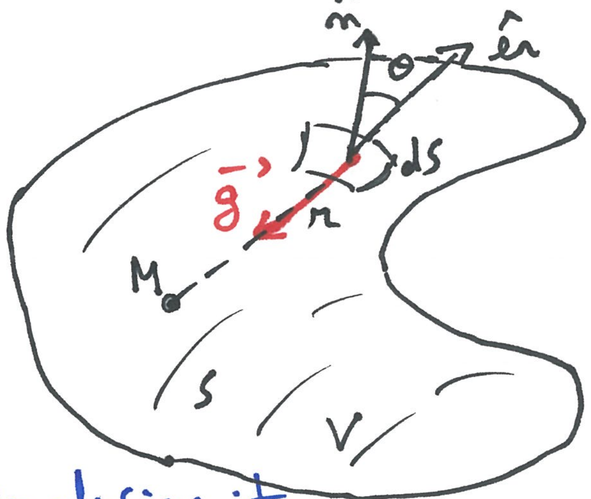
- Understood that was universal and in the Principia drew the figure:



• Now $\vec{F} = -\frac{GMm}{r^2}$, i.e., $\vec{g} = -\frac{GM}{r^2}\hat{e}_r$ is the field equation!

It is better to have a differential equation for $\vec{g}$ or for ϕ.

Let us see this with the help of the figure.



• Study the gravitational field due to a point mass M. Consider volume V enclosing it. The vector $\hat{n}$ is the normal to the surface S enclosing V.

Compute the surface integral $I = \int_S \hat{n}\cdot\vec{g}\, dS$ the flux of $\vec{g}$ through S.

Check that $r^2 d\Omega = \cos\theta\, dS$. Also $\hat{n}\cdot\vec{g} = -\frac{GM}{r^2}\cos\theta$

Then $I = \int_S -\frac{GM}{r^2}\cos\theta \frac{r^2 d\Omega}{\cos\theta} = -GM\int_S d\Omega = -4\pi GM$

• For several masses $I = -4\pi G \sum_{\text{each } M} M = -4\pi G M_{total} = -4\pi G \int_V \rho\, dV$

Gauss: $\int \hat{n}\cdot\vec{g}\, dS = \int_V \vec{\nabla}\cdot\vec{g}\, dV$. So $\int_V (\vec{\nabla}\cdot\vec{g} + 4\pi G\rho)\, dV = 0 \Longrightarrow$

$\vec{\nabla}\cdot\vec{g} = -4\pi G\rho$. the Field equation

But $\vec{g} = -\vec{\nabla}\phi \Longrightarrow \nabla^2\phi = 4\pi G\rho$ is Poisson equation.

- There is another equation as important: the deviation equation of two particles

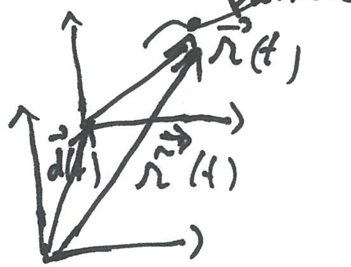
- To find it recall the equation of motion for one particle in a gravitational field $\vec{g}(\vec{r})$: $m_{in}\,\vec{a} = \vec{F}$, i.e.,

$$m_{in} \frac{d^2\vec{r}(t)}{dt^2} = m_{grav}\, \vec{g}(\vec{r}(t))$$

From experiments of Newton, Eötvos, Dicke, Braginsky: $m_{in} = m_{grav}$

so equation is $$\frac{d^2\vec{r}(t)}{dt^2} = \vec{g}(\vec{r}(t))$$

- We can change reference frame, $t \to t$ and $\vec{\tilde{r}}(t) = \vec{r}(t) + \vec{d}(t)$



Then $$\frac{d^2\vec{\tilde{r}}(t)}{dt} = \vec{g}(\vec{\tilde{r}}(t)) + \frac{d^2\vec{d}(t)}{dt^2}$$

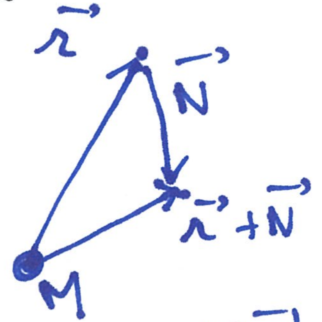
choosing $\vec{d}(t)$ appropriately one can eliminate gravity $\vec{g}$ along the path the particle follows.

Exercise: Find $\vec{d}(t)$ in the Earth's uniform gravitational field.

- Equivalence principle: All objects in free fall follow the same path if they have the same initial velocity.

- Since $\vec g$ depends on $\vec r$ we can distinguish between a real gravitational field and a false one. Example of a false field: a constant acceleration field.

- Have to look for the differential motion between particles to distinguish them: this implies tidal forces



- for particle in $\vec r$: $\dfrac{d^2 \vec r}{dt^2} = \vec g(\vec r, t)$

  for particle in $\vec r + \vec N$: $\dfrac{d^2\, \vec r + \vec N}{dt^2} = \vec g(\vec r + \vec N, t)$

subtracting

$$\frac{d^2 \vec N}{dt^2} = \vec g(\vec r + \vec N, t) - \vec g(\vec r, t)$$

or $$\frac{d^2 \vec N}{dt^2} = \cancel{g(\vec r, t)} + \vec N \cdot \vec\nabla \vec g - \cancel{\vec g(\vec r, t)}$$

$$\frac{d^2 \vec N}{dt^2} = (\vec N \cdot \vec \nabla) \vec g \quad \text{in first order}$$

- so $\dfrac{d^2 N_i}{dt^2} = \left(N_j \dfrac{\partial}{\partial x^j}\right) g_i$ or $\dfrac{d^2 N_i}{dt^2} = \dfrac{\partial g_i}{\partial x^j} N_j$ define $M_{ij} = -\dfrac{\partial g_i}{\partial x^j}$

$\Rightarrow$ Eq. for the separation is $\dfrac{d^2 N_i}{dt^2} = -M_{ij} N_j$ or $\dfrac{d^2 N_i}{dt^2} + M_{ij} N_j = 0$

This is the Newton deviation equation or tidal equation.

- Property of the tidal tensor: since $\vec{g} = -\vec{\nabla}\phi \longrightarrow \vec{\nabla}\times\vec{g} = 0$

or in index notation $\varepsilon_{ijk}\,\partial_j g_k = 0$ $\qquad \varepsilon_{ijk}: \begin{cases} \varepsilon_{123} = 1 \\ \varepsilon_{112} = 0 \end{cases}$ and so on

Then $\underset{\text{(no sum on } i)}{\varepsilon_{ipq}\,\varepsilon_{ijk}\,\partial_j g_k = 0} \Longrightarrow \left(\delta_{pj}\cdot\delta_{qk} - \delta_{pk}\,\delta_{jk}\right)\partial_j g_k = 0$

so $\partial_p g_q - \partial_q g_p = 0 \Longrightarrow M_{qp} = M_{pq}$ or $M_{ij} = M_{ji}$ is symmetric

M_{ij} is the tidal tensor, important in Newtonian gravitation.

- How to write the field equation $\vec{\nabla}\cdot\vec{g} = -4\pi G\rho$. In index

$\partial_i g_i = -4\pi G\rho$ or $M_{ii} = 4\pi G\rho$ is the field equation or the Poisson eq.

- Yet another equation for M_{ij}: since $M_{ij} = -\frac{\partial g_i}{\partial x^j} \Longrightarrow$

$\partial_k M_{ij} = -\partial_k \partial_j g_i = -\partial_j \partial_k g_i = \partial_j M_{ik}$ so $M_{ij,k} - M_{ik,j} = 0$

or $M_{i[j,k]} = 0$ Bianchi identity in Newtonian gravitation.

- Newtonian gravitation equations in brief:

(1) $M_{ii} = 4\pi G\rho$ Field equation (Poisson) $\qquad$ (2) $\frac{d^2N^i}{dt^2} + M_{ij}N_j = 0$ Deviation equation

(3) $M_{ij} = M_{ji}$ symmetry equation $\qquad$ (4) $M_{i[j,k]} = 0$ Bianchi identity

- Exercise: Find M_{ij} for a spherically symmetric field.

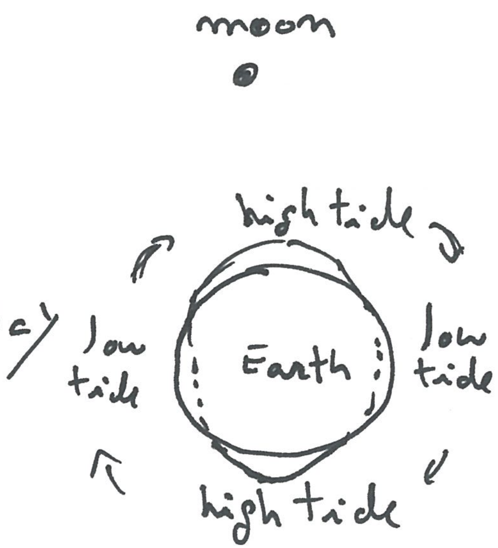
- Consequences of Newtonian gravitation

Motion of planets solved. The 3 empirical Kepler laws explained.

Masses of the planets obtained.

Discovery of new planets.

Tides first ever explained: two tides per day



Technology: artificial satellites

Galileo's problem solved: acceleration is the same for all bodies in the gravitational field of the Earth, $a = g$.

Pendulum's period independent of the mass, $T = 2\pi\sqrt{\frac{\ell}{g}}$.

The weight of a body as a function of latitude: centrifugal force

3. General Relativity

3.1 Special relativity and Minkowski spacetime

- Maxwell showed that c was the electromagnetic velocity relative to some ether. It was also light velocity. In a stroke unified electricity, magnetism and opticks.

- Einstein in 1905 turned Galilean relativity into special relat. (i) All physics is the same for any inertial observer (ii) The speed of light c is the same for all observers. It is a universal constant.

- Minkowski finalized the logic and mathematical expression. Space and time were unified in spacetime with the invariant interval

$$ds^2 = g_{ab}\,dx^a dx^b = -dt^2 + dx^2 + dy^2 + dz^2,$$

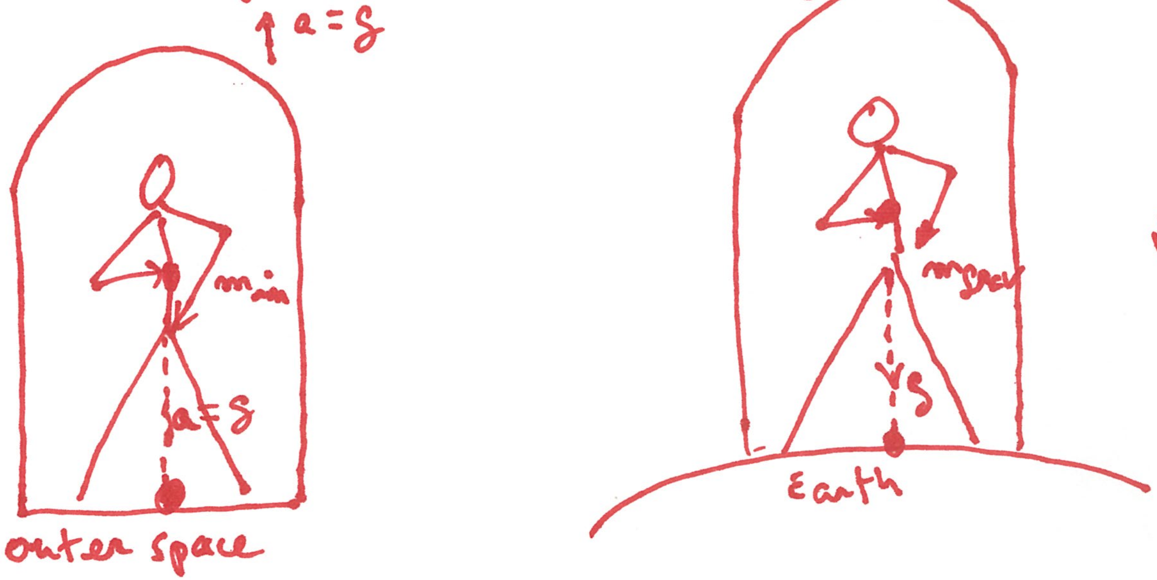
$g_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is the metric, Minkowski metric in flat coordinates.

3.2 The equivalence principle

- From experiments $m_{in} = m_{grav}$ withow any reason. Newton was puzzled.

- Einstein incorporated this equality into the equivalence principle. The idea is that it is not possible to distinguish the 2 masses.

Elevator experiments:



- Equivalence principle: in the vicinity of any given point a gravitational field is equivalent to an accelerated reference frame. Thus, $m_g \equiv m_i$.

3.3 Particles, geodesics, and accelerated trajectories

- No gravitational field $\Longrightarrow$ spacetime is Minkowski, it is flat.

  Flat spacetime : geodesics are straight lines $\frac{d^2x^a}{d\tau^2} = 0$
  τ propertime.

  And particles follow geodesics! If there are no forces.

- On the other hand, accelerated particles should obey

$$\frac{d^2x^a}{d\tau^2} + \Gamma^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = 0$$

  where $\Gamma^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau}$ represent inertial accelerations: centrifugal, coriolis, and others. Moreover $\Gamma^a_{bc} \simeq g_{ab,c} + \dots$

  g_{ab} metric in general coordinates.

- From equivalence principle acceleration = gravitation, locally.
  It is also equation for a trajectory in a gravitational field.

- So in $\frac{d^2x^a}{d\tau^2} + \Gamma^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = 0$ 1st term is acceleration
2nd term is gravitational force. Since $\Gamma^a_{bc} = g_{ab,c}$
and $g_i = \partial_i \phi \implies g_{ab}$ are the gravitational potentials.
Since $M_{ij} = \partial_i \partial_j \phi \implies g_{ab,cd}$ are the tidal components

- But spacetime is geometry. So everything should be put in geometric language: Riemannian geometry the metric tensor g_{ab} is the fundamental tensor.

- In brief:

Newtonian gravitation	General relativity
ϕ	g_{ab}
$F_i = \partial_i \phi$	$\Gamma^a_{bc} = g_{ab,c} + \dots$
$M_{ij} = \partial_i \partial_j \phi$	$R^a{}_{bcd} = g_{ab,cd} + \dots$
	Riemann tensor

• Recall Newtonian gravitation equation (1) Poisson equation

$$M_{ii} = 4\pi G\rho \qquad \text{or } \nabla^2\phi = 4\pi G\rho$$

Einstein generalized to the field equation

$$G_{ab} = 8\pi \frac{G}{c^4} T_{ab}.$$

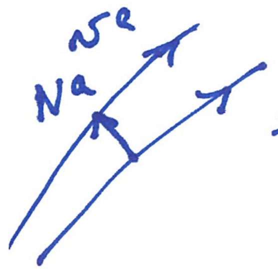
G_{ab} is essentially Riemann tensor $R^a{}_{bcd}$ contracted, generalizes M_{ii} tidal tensor M_{ij} contracted. T_{ab} is the energy-momentum tensor.

• Recall Newtonian gravitation equation (2) Deviation equation

$$\frac{d^2 N^i}{dt^2} + M^i{}_j N_j = 0$$

Einstein generalized to the geodesic deviation equation

$$\frac{d^2 N^a}{d\tau^2} + R^a{}_{bcd} V^a V^b N^d = 0.$$



Deviation of two geodesics

- Equation (3) was $M_{[ij]} = 0$ now is $R^a{}_{[bcd]} = 0$.

  Equation (4) was $M_{i[j,k]} = 0$ now is $R^a{}_{b[cd,e]} = 0$ Bianchi identity

- The important equation is the Einstein equation

$$G_{ab} = \frac{8\pi G}{c^4} T_{ab}$$

curvature of spacetime = matter energy

spacetime tells matter how to move. Matter tells spacetime how to curve.

Note the fundamental constants G and c appear naturally.

3.5 Consequences of general relativity

- Classical tests: Mercury perihelion precession, light deflection in the gravitational field of the sun and lensing, gravitational Doppler effect, delay in the radar echo from a planet.
- Technological applications: GPS. Would not function without GR.

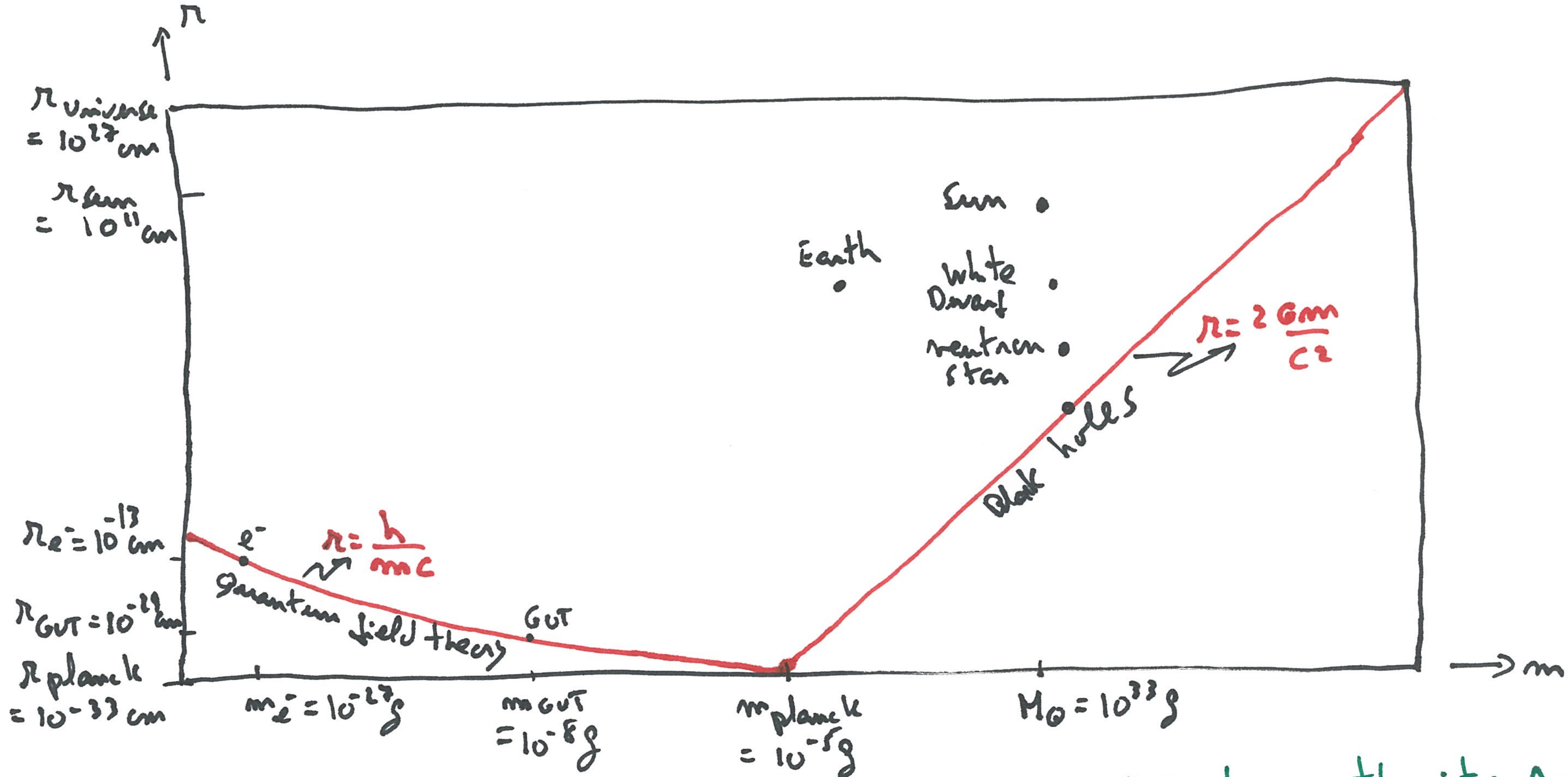
- Gravitational waves: spacetime waves predicted by Einstein 1916 Detected 2015. Nobel prize 2017.

- Cosmology: dynamical study of the universe. Started by Einstein in 1917. Then expanding universe of Friedmann, Lemaître, Hubble.

- Black holes: the pure geometric object of GR par excellence. Oppenheimer and Snyder 1939 discovered it. Nobel prize to Penrose 2020. Einstein never understood it.
- Fundamental theories: Unification of gravitation, electromagnetism and other fields. Started by Weyl 1918 continuing up to now.

4. Quantum Gravity

4.1 Length scales and mass scales in physics



In physics 2 formulas relate mass of an object with its radius.

quantum: $r_{compt} = \frac{h}{mc}$ general relativity $r_{schw} = \frac{2Gm}{c^2}$

meet when $r_{compt} = r_{schw} \Rightarrow$ $r_{planck} = \sqrt{\frac{G\hbar}{c^3}} = 10^{-33}$ cm $m_{planck} = \sqrt{\frac{\hbar c}{G}} = 10^{-5}$ g

4.2 Feynman diagrams

- Quantum electrodynamics : two basic ingredients

e⁻ line
photon line

The diagrams show processes in space and time where particles travel from one place to the other and can interact. Possible processes are constructed connecting lines of e^- and of photons.

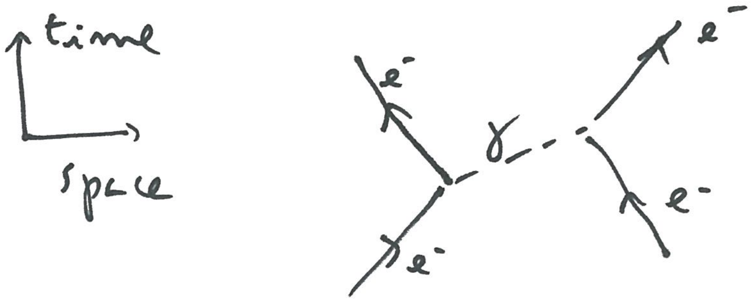
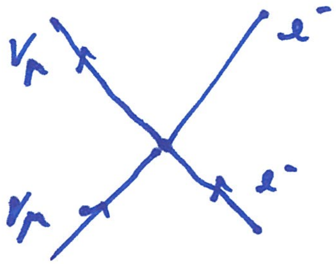


diagram of e^-e^- interaction with photon exchange

Each vertice contributes with $\alpha = \frac{e}{\hbar c} = \frac{1}{137} \ll 1$. So expansion in α possible.

- Electroweak interactions

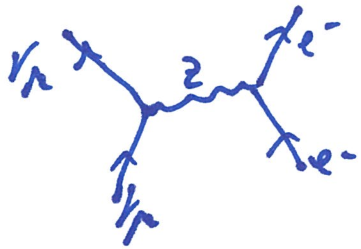
Fermi theory



$\nu_\mu e^-$ scattering

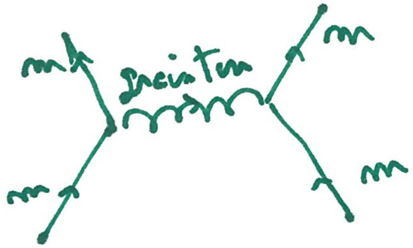
Each vertice contributes with $\alpha_F E^2$ which diverges for high E

Weinberg - Salam theory



New theory: the four-Fermi interaction is resolved in the exchange of the vector boson Z

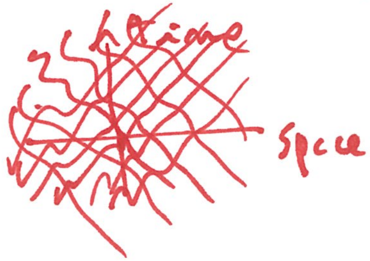
• Gravitational interaction: gravitons



exchange of a graviton between two particles. Each vertice contributes with GE^2 and diverges as E is high!

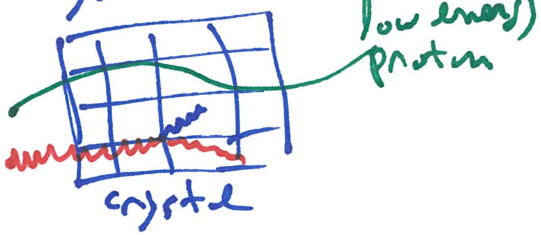
How to cure these divergences? obtain a proper quantum gravity theory.

• Gravitation and curved spacetime are the ~~same~~ different words for the same thing. So admit that spacetime is a construct of a host of gravitons



In low energy physics one graviton can travel through spacetime, i.e., through the host of gravitons.

• Like a low energy proton interacting with a crystal. It doesn't see the structure of the crystal.

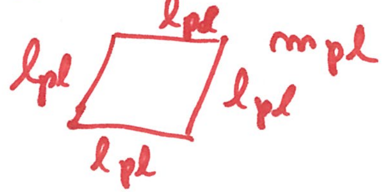


whereas a high energy proton interacts with the protons of the crystal

- In the same manner: a very high energy gravitons interacts with the gravitons that make spacetime. Quantum gravity is a theory about those gravitons.

- Since quantum mechanics and gravitation meet at the Planck scale the basic structure reveals itself at that scale.

Planck lengths and Planck masses are the scales involved in the structure.



It can be a foam structure, a stringy-brane structure, a knotty structure. You pick the one that you think leads to the correct theory.

COSMIC OUROBOROS

{ Minimum length 10^{-33} cm
= Planck length