9th CENTRA School on Astrophysics and Gravitation

Gravitational Waves: Theory and Experiment

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These notes where prepared for 4×50 minute lectures delivered at the 9^{th} CENTRA school on astrophysics and gravitation: `https://centra.tecnico.ulisboa.pt/network/eag9/`.

The following textbooks were used during the preparation of these notes:

1. Michele Maggiore, "Gravitational Waves. Volume 1: Theory and Experiment". Oxford University Press; 2008.

2. Bernard F. Schutz, "A first course in general relativity". Cambridge University Press; 1985.

Chapter I of these notes, covering the material in the first two lectures, was prepared by David Hilditch and chapter II, covering the final two lectures, was prepared by Christopher J. Moore. If you spot any typos in these note, or have comments or questions on the material covered in the lectures then please contact `david.hilditch@tecnico.ulisboa.pt` or `christopher.moore@technico.ulisboa.pt`.

Contents

Chapter 1

Theory

1.1 Recap of General Relativity

Let us introduce *coordinates* $x^\mu = (ct, x^i) = (ct, x, y, z)$. Throughout we will take Greek indices $\mu = 0 \ldots 3$, and Latin indices $i = 1 \ldots 3$. The *metric* is an invertible symmetric matrix $g_{\mu\nu}$, each component of which is a function of the coordinates; i.e. $g_{\mu\nu}(x)$. "Symmetric" here means that $g_{\mu\nu} = g_{\nu\mu}$. The metric is taken to have Lorentzian signature, which means that it has one negative and three positive eigenvalues. The metric allows us to measure distances and times; for two points in spacetime separated by some infinitesimal change in coordinates $\mathrm{d}x^\mu$ the proper distance interval between these points is defined to be

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \,. \tag{1.1}$$

The inverse metric is denoted by $g^{\mu\nu}$. We use the *Einstein summation convention*, which means that repeated indices, one upstairs and one downstairs, are understood to be summed over. In other words the expression $g^{\mu\alpha}g_{\alpha\nu}$ corresponds simply to matrix multiplication and so by definition we have $g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu{}_\nu$.

In General Relativity (GR) the metric replaces the Newtonian gravitational potential as the unknown, and is required to satisfy *Einstein's equations*:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \,. \tag{1.2}$$

The object on the right hand side, $T_{\mu\nu}$, is called the *stress-energy tensor*. G and c are Newton's constant and the speed of light, respectively. The stress-energy tensor is symmetric in $\mu\nu$, and encodes the local energy density, linear momentum and the stresses and strains of any matter content. The object on the left hand side is called the *Einstein tensor*. It is also symmetric, and is constructed from the *Ricci tensor* $R_{\mu\nu}$ as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \,, \tag{1.3}$$

with the *Ricci scalar* $R = g^{\mu\nu} R_{\mu\nu}$. A general feature of our notation is that, unless doing so would clash with any previously defined quantity, if the inverse metric is used to trace over two indices of a matrix the resulting object gets the same name but without the indices. The Ricci curvature is in turn defined as a trace of the *Riemann curvature* $R_{\mu\nu\rho}{}^{\sigma}$,

$$R_{\mu\nu} = R_{\mu\rho\nu}{}^{\rho} \,, \tag{1.4}$$

$$R_{\mu\nu\rho}{}^{\sigma} = \partial_\nu \Gamma^{\sigma}{}_{\mu\rho} - \partial_\mu \Gamma^{\sigma}{}_{\nu\rho} + \Gamma^{\alpha}{}_{\mu\rho}\Gamma^{\sigma}{}_{\nu\alpha} - \Gamma^{\alpha}{}_{\nu\rho}\Gamma^{\sigma}{}_{\mu\alpha} \,, \tag{1.5}$$

where the *Christoffel symbols* are given by first derivatives of the metric,

$$\Gamma^{\gamma}{}_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta}(\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta}) \,. \tag{1.6}$$

With this soup of indices and derivatives it is easy to get lost, but the take home message is that the left-hand side of the field equations contains objects related to the curvature, and the right-hand side stuff to do with matter. This is well-desribed by Wheeler's famous statement that "Spacetime tells matter how to move; matter tells spacetime how to curve". What's more, since the curvature contains one derivative of the Christoffel, and the Christoffel contains one derivative of the metric, we have

$$G_{\mu\nu}[\partial\partial g, \partial g, g] \approx T_{\mu\nu}[\text{Matter}] \,. \tag{1.7}$$

just like in Newtonian gravity which has Poisson's equation;

$$\nabla\phi = 4\pi G \rho \,. \tag{1.8}$$

It is important to realize however the *crucial* difference that in the latter case the field equations are linear, whereas in GR they are not.

1.2 Linearized Field Equations (with matter)

To try and understand basic properties of solutions to the field equations (1.3) we can consider small perturbations of a known solution. In the absence of any nontrivial gravitational field we can introduce coordinates in which the metric becomes,

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{1.9}$$

just like in special relativity. Therefore let us now consider solutions with a metric of the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $h_{\mu\nu}$ small. The inverse metric can then be written as,

$$g^{\mu\nu} = \eta^{\mu\nu} - \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta} + O(h^2). \tag{1.10}$$

Henceforth we will allow the *background metric* $\eta^{\mu\nu}$ to raise and lower indices, so that $h^{\mu\nu} \equiv \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$. Plugging this ansatz into the field equations and keeping only terms linear in $h_{\mu\nu}$ we obtain, the linearized Einstein equations,

$$-\frac{1}{2}\Box\bar{h}_{\mu\nu} + \partial^{\lambda}\partial_{(\mu}\bar{h}_{\nu)\lambda} - \frac{1}{2}\eta_{\mu\nu}\partial^{\alpha}\partial^{\beta}\bar{h}_{\alpha\beta} = \frac{8\pi G}{c^4}T_{\mu\nu}\,. \tag{1.11}$$

with $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$, the wave, or d'Alembert, operator is denoted by $\Box \equiv \eta^{\alpha\beta}\partial_\alpha\partial_\beta$ and round brackets around two indices denotes symmetrization $X_{(\mu\nu)} \equiv (X_{\mu\nu} + X_{\nu\mu})/2$. For self-consistency of the approximation we must now assume that the stress-energy tensor is small. On the left-hand side we see that the first term is of a simple form that often appears elsewhere. Solutions to that operator can be treated in a standard way by using Green's functions. Unfortunately the second and third terms are more complicated. We might therefore wish they were absent.

Exercise 1.1: The linearized inverse metric
Verify that, up to order h, the linearized inverse metric is indeed given by (1.10).

1.3 Gauge Freedom

Consider the linearized Riemann curvature (1.5),

$$2R_{\mu\nu\lambda\rho} = \partial_\mu\partial_\rho h_{\lambda\nu} - \partial_\mu\partial_\lambda h_{\rho\nu} - \partial_\nu\partial_\rho h_{\lambda\mu} + \partial_\nu\partial_\lambda h_{\rho\mu}\,, \tag{1.1}$$

for a metric perturbation of the form $h_{\alpha\beta} = \partial_\alpha\phi_\beta + \partial_\beta\phi_\alpha$ we have $R_{\mu\nu\lambda\rho} = 0$.

Since the field equations are built from contractions of the curvature, this means that we have a 'gauge-freedom', like the freedom to add a constant to the gravitational potential or (more closely) like in Electromagnetism. Let us exploit this to see if we can make our ugly equations (1.11) pretty. To do so we modify the metric perturbation by with a 'gauge change' of the form,

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} - \partial_{(\mu}\xi_{\nu)}. \tag{1.2}$$

By the observation above, this will take solutions of the field equations to solutions. Physically the two should then be identified. Choosing the *Lorenz-gauge*

$$\Box\xi_\alpha = \partial^\beta\bar{h}_{\alpha\beta}\,, \tag{1.3}$$

we find that the modified metric perturbation must satisfy,

$$\Box\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu}\,, \tag{1.4}$$

$$\partial^\nu\bar{h}_{\mu\nu} = 0\,. \tag{1.5}$$

This is much like Lorenz-gauge in Electromagnetism. We have succeeded in getting rid of all of the messy looking terms in equation (1.11)!

In fact when we are in vacuum we can do even better, by noting that if we make yet another adjustment, but this time taking $\Box\bar{\xi}_\mu = 0$, our solution will still still satisfy the Lorenz-gauge condition. Using this we can arrive at the *Transverse-Traceless* (TT) gauge

$$h = 0\,, \qquad h_{0j} = 0\,. \tag{1.6}$$

To achieve this we just choose $\bar{\xi}_\mu$ such that,

$$\partial_\mu\bar{\xi}^\mu = \frac{1}{2}h\,, \qquad \partial_0\bar{\xi}_j + \partial_j\bar{\xi}_0 = h_{0j}\,. \tag{1.7}$$

By imposing the additional four equations (1.6) we have used up the remaining gauge freedom. Threrefore we expect that there are

$$2 = 10\ \#\ \text{(Field equations)} - 4\ \#\text{(Lorenz gauge conditions)} - 4\ \#\text{(TT gauge conditions)}$$

remaining physical degrees of freedom; these are the *gravitational waves* (GWs)!

Exercise 1.2: Curvature of a pure gauge metric perturbation
Show that the linearized Riemann tensor (1.1) vanishes for a metric perturbation of the form $h_{\alpha\beta} = \partial_\alpha\phi_\beta + \partial_\beta\phi_\alpha$. Hint: substitute the perturbation into the expression and use equality of mixed partials.

Exercise 1.3: Linearized Gravity in Lorenz-gauge
Starting with the stated gauge adjustment, show that equations (1.4) and (1.5) hold true in the Lorenz gauge. You may use the expressions,

$$R_{\mu\lambda} = \partial^\nu\partial_{(\mu}h_{\lambda)\nu} - \frac{1}{2}\Box h_{\lambda\mu} - \frac{1}{2}\partial_\mu\partial_\lambda h\,, \tag{i}$$

$$R = \partial^\nu\partial^\lambda h_{\nu\lambda} - \Box h\,, \tag{ii}$$

for the linearized Ricci tensor and scalar. As in exercise 1.2 the important fact to use is the equality of mixed partial derivatives.

1.4 Plane wave solutions

Let us work in vacuum and make a plane wave ansatz

$$h_{\mu\nu} = \Re\left[e_{\mu\nu} e^{ik_\lambda x^\lambda}\right] . \tag{1.1}$$

When computing we will just take the real part at the end. The matrix $e_{\mu\nu}$ is constant in space and time. It is called the polarization tensor. The vector k_μ is called the wave vector. Plugging both this ansatz into the Lorenz-gauge Einstein equations we obtain,

$$k^\mu k_\mu [\text{ Nonvanishing Stuff }] = 0, \tag{1.2}$$

so that the wave-vector is null $k^\mu k_\mu = 0$. Physically this means that the solutions all travel at light-speed, which is not surprising since the field equations have been reduced to the wave equation. Not all of these traveling solutions represent physical waves however, as we must still impose the Lorenz-gauge. Substituting our ansatz into (1.5) we obtain,

$$k^\nu e_{\mu\nu} = \frac{1}{2} k_\mu e^\nu{}_\nu , \tag{1.3}$$

as expected restricting four of the ten components. Suppose now that the wave is propagating in the z direction, then $k^\mu = (k, 0, 0, k)$. In TT gauge $e^\mu{}_\mu = 0$, so it follows that $e^\mu{}_\nu k^\nu = 0$. In TT gauge we also have $e_{0j} = 0$ and hence $e_{0\mu} = e_{3\mu} = 0$. The polarization tensor can thus be written,

$$e_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{1.4}$$

The polarization tensor is then a sum over two independent matrices, one associated with the constant h_+, the other with $h_\times$, these are the two gravitational wave polarization states. This will be discussed in more detail in the following lecture.

1.5 The generation of gravitational waves

We have seen that in Lorenz gauge the linearized Einstein equations with sources are,

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu} , \tag{1.5}$$

$$\partial^\nu \bar{h}_{\mu\nu} = 0 . \tag{1.6}$$

Working in complete generality by assuming slowly moving, weak, weakly self-gravitating, isolated sources, and evaluating the solution far from the source, the TT part of the metric perturbation can be written as,

$$h_{kl}^{\mathrm{TT}} \approx \frac{2G}{c^4 r} \perp_{kl}^{ij}(n)\, \ddot{\mathcal{I}}_{ij}(t - r/c)\,. \tag{1.7}$$

This is the famous *quadrupole formula*. It determines the gravitational wave emission from the traceless mass quadrupole,

$$\mathcal{I}_{ij}(t) = \int_{\mathbb{R}^3} \rho(t, x'^i)\big(x'_i x'_j - \frac{1}{3}\delta_{ij} r'^2\big) d\vec{x}'\,. \tag{1.8}$$

The isolated system is assumed to be located near the origin. The radial scalar function is defined in the obvious way as $r^2 = x^2 + y^2 + z^2$. The vector n^i is the unit spatial vector in the direction of propagation of the gravitational wave. Since our observer is assumed to lie a large distance from the source, we can set $n^i = x^i/r$. Finally the TT projection operator is,

$$\perp_{kl}^{ij}(n) = q^i{}_{(k} q^j{}_{l)} - \frac{1}{2} q_{kl}\, q^{ij}\,, \tag{1.9}$$

$$q^i{}_j = \delta^i{}_j - n^i n_j\,. \tag{1.10}$$

Here we raise and lower indices with the kronecker delta δ^{ij}, which corresponds to the spatial part of the background metric. The derivation of the quadrupole formula starts with the Green's function form of the solution to the wave equation. The resulting expression is then manipulated using the simplifying assumptions stated above.

Box 1.1: Gravitational waves from a circular Newtonian binary system

Let us assume that our matter source consists of a binary system consisting of two point particles of equal mass traveling on Newtonian circular orbits in the xy plane, and compute the TT part of the metric perturbation for an observer on the positive z-axis. Those of you who have studied GR may note that this exercise is rather artificial, because the point particles should follow geodesics of flat-space, and should hence not travel on a circular orbit. This shortcoming can be fixed within the Post-Newtonian approximation. This example nevertheless serves as an important first computation.

The positions of the two masses can then be written,

$$x_1^i(t) = R(\cos(\omega t), \sin(\omega t), 0)\,, \quad x_2^i(t) = R(-\cos(\omega t), -\sin(\omega t), 0)\,. \tag{i}$$

The radius of the orbit R is a constant that can be computed from Kepler's third law as $R^3 = GM/\omega^2$ with the total mass of the system M.

The mass-density for this system is given by,

$$\rho = \frac{1}{2} M\delta(x - x_1, y - y_1, z) + \frac{1}{2} M\delta(x - x_2, y - y_2, z)\,, \qquad \text{(ii)}$$

with δ here denoting the three-dimensional Dirac delta function. Using some trigonometric identities the traceless mass quadrupole can then be computed as,

$$\mathcal{I}_{ij} = \frac{1}{8} MR^2 \begin{pmatrix} \cos(2\omega t) + \frac{1}{3} & \sin(\omega t) & 0 \\ \sin(2\omega t) & -\cos(2\omega t) + \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}. \qquad \text{(iii)}$$

The TT projection operator $\perp^{ij}_{kl}(n)$ is easily constructed using the fact that for our observer $n^i = (0, 0, 1)$, so that,

$$q^i{}_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad \text{(iv)}$$

In fact however, looking at the form of the traceless mass quadrupole we see that in this particular case we need not apply the TT projection, since the second derivative of $\mathcal{I}_{ij}$ will already be transverse to z and traceless in the xy-plane already. Again using Kepler's third law and putting this all together we find that the TT metric perturbation is,

$$h^{TT}_{kl} = -\frac{GM}{c^4 r} (GM\omega)^{2/3} \begin{pmatrix} \cos(2\omega t) & \sin(2\omega t) & 0 \\ \sin(2\omega t) & -\cos(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad \text{(v)}$$

The important point to realize here is that the gravitational wave frequency is twice the orbital frequency. This is a reflection of the fact that GR is a spin-2 theory.

Exercise 1.4: Revisiting the Newtonian binary
Carefully reproduce the calculation presented in Box 1.1, checking all of the factors throughout. What does the TT metric perturbation look like for observers lying on the x-axis?

1.6 Energy loss via gravitational waves

It is not trivial to establish, but nevertheless true that gravitational waves carry energy. The difficulty here stems from the fact that the large gauge freedom of even the linearized field equations makes it hard to untangle that which is physical from that which is not. There is however a notion of gravitational wave stress-energy due to Isaacson, who defines,

$$T_{\mu\nu} = \frac{c^4}{32\pi G}\left\langle \partial_\mu h_{ij}^{\mathrm{TT}} \partial_\nu h^{ij}_{\mathrm{TT}} \right\rangle . \tag{1.1}$$

The wedge brackets here denote that an average must be taken over a few gravitational wavelengths. The idea of this is that whilst a gauge transformation can eradicate a physical wave at a point, it can not do so over an extended region.

The energy radiated in gravitational waves (the GW luminosity) is determined by integrating the flux of energy momentum over a sphere of large radius,

$$L_{\mathrm{GW}} = \frac{\mathrm{d}E_{\mathrm{GW}}}{\mathrm{d}t} = -\int \mathrm{d}A\, T_{0j}^{\mathrm{GW}} n^j = \int \mathrm{d}A\, T_{00}^{\mathrm{GW}} . \tag{1.2}$$

In the quadrupole approximation this becomes,

$$L_{\mathrm{GW}} = \frac{G}{5c^5}\left\langle \dddot{\mathcal{I}}_{ij} \dddot{\mathcal{I}}^{ij} \right\rangle . \tag{1.3}$$

So the system of the worked example in Box 1.1 above should slowly lose energy. The standard picture is that this emission causes the binary orbits to contract so that the objects eventually merge! Throughout the inspiral the approximations that we have used to describe the physics here gradually breakdown, and must be replaced with more sophisticated methods, such as the Post-Newtonian approximation mentioned above, and close to the time of merger full numerical relativity.

Chapter 2

Experiment

The first part of this course introduced gravitational waves (GWs) as weak perturbations in the gravitational field propagating through spacetime at the speed of light. In order to detect GWs we must build some kind of instrument which will interact with a passing GW. (For example, an antenna can be used the detect radio waves because the negatively charged electrons in the wire interact with the oscillating electric field in a passing electromagnetic wave.) The purpose of this second part of this course is to describe, at a very idealised level, how GWs interact with 2 very different types of instrument: resonant bars, and interferometers. To begin with, we will consider an idealised GW detector built from a number of non-interacting, freely falling test particles. Therefore, our starting point will be the equation of motion for a test particle in general relativity.

2.1 The Geodesic Equation

In general relativity a freely falling test particle moving in a curved background spacetime follows a *timelike geodesic* in the spacetime metric, $g_{\mu\nu}(x)$. Using a system of coordinates x^{μ}, the *worldline* of a test particle is the curve $x^{\mu}(\lambda)$ [see figure (2.1)]. The parameter λ labels points along the curve and is otherwise completely arbitrary; we are free to choose any convenient parameterisation. For two nearby points on the curve, $x^{\mu}(\lambda)$ and $x^{\mu}(\lambda+\mathrm{d}\lambda)$, the distance interval between them is calculated using the metric;

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\,\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda}\,\frac{\mathrm{d}x^{\nu}}{\mathrm{d}\lambda}\,\mathrm{d}\lambda^2\,. \tag{2.1}$$

We will assume that the curve is *timelike*, meaning $\mathrm{d}s^2 < 0$. (Remember, we are using the Minkowski metric sign convention $\eta_{\mu\nu} = \mathrm{diag}[-,+,+,+]$.) The *proper time* interval along the worldline is defined as

$$\mathrm{d}\tau^2 = \frac{-\mathrm{d}s^2}{c^2}\,. \tag{2.2}$$

The proper time is the time that would be measured by a clock moving along the curve $x^\mu(\lambda)$. Because $\mathrm{d}s^2 < 0$ the proper time is always increasing along the curve. Therefore, it is natural to use τ as a parameter along the curve; we write the position of the particle as a function of its own proper time, $x^\mu(\tau)$. The particle's *four-velocity* is defined as $u^\mu(\tau) = \mathrm{d}x^\mu/\mathrm{d}\tau$. Notice, that equation (2.1) implies that the magnitude of the four-velocity vector is constant along the worldline of the particle;

$$-c^2 = g_{\mu\nu}(x)\, u^\mu(\tau)\, u^\nu(\tau)\,. \tag{2.3}$$

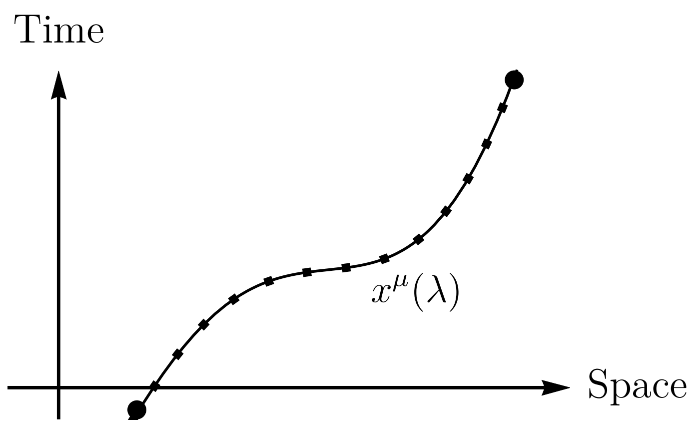

Figure 2.1: One possible *worldline* for a test particle through spacetime linking two fixed endpoints. The parameter λ labels points on the curve and is otherwise arbitrary; the tick marks along the curve are intended to indicate successive intervals of λ.

Among all possible curves linking two fixed endpoints, $x^\mu(\tau_A)$ and $x^\mu(\tau_B)$, a test particle will follow the *geodesic* which extremises the total proper time along the curve. The total proper time is given by

$$S\big[x^\mu(\tau)\big] = \int_{\tau_A}^{\tau_B} \mathrm{d}\tau = \int_{\tau_A}^{\tau_B} \frac{\mathrm{d}\tau}{c}\, \sqrt{-g_{\mu\nu}(x)\, u^\mu(\tau)\, u^\nu(\tau)}\,. \tag{2.4}$$

This may remind you of the principle of least action from classical mechanics. The calculus of variations can be used to find the equation that must be satisfied by the curve (see box 2.1) and the resultant geodesic equations are

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu{}_{\nu\rho}(x) \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} = 0\,, \tag{2.5}$$

$$\text{or} \quad \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu{}_{\nu\rho}(x)\, u^\mu\, u^\nu = 0\,, \tag{2.6}$$

where the following quantities (know as the *connection coefficients*, or sometimes as the *Christoffel symbols*) have been defined,

$$\Gamma^{\mu}{}_{\nu\rho}(x) = \frac{1}{2} g^{\mu\alpha}(x) \big[\partial_\nu g_{\alpha\rho}(x) + \partial_\rho g_{\alpha\nu}(x) - \partial_\alpha g_{\mu\nu}(x) \big] \,. \tag{2.7}$$

Equations (2.5) or (2.6) are the equations of motion for a test particle in general relativity.

Box 2.1: The geodesic equation from calculus of variations

The integrand, or *Lagrangian*, of the functional in equation (2.4) is given by

$$\mathcal{L}\big(x^\mu, u^\mu\big) = \sqrt{-g_{\mu\nu}(x)\, u^\mu(\tau)\, u^\nu(\tau)} \,. \tag{i}$$

The Euler-Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial \mathcal{L}}{\partial u^\mu}\right) = \frac{\partial \mathcal{L}}{\partial x^\mu} \,. \tag{ii}$$

Evaluating the partial derivatives gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{g_{\mu\nu}(x)\, u^\nu(\tau)}{\sqrt{-g_{\alpha\beta}(x)\, u^\alpha(\tau)\, u^\beta(\tau)}}\right) = \frac{\frac{1}{2}\partial_\mu g_{\gamma\delta}(x)\, u^\gamma(\tau)\, u^\delta(\tau)}{\sqrt{-g_{\alpha\beta}(x)\, u^\alpha(\tau)\, u^\beta(\tau)}} \,. \tag{iii}$$

Using the result in equation (2.3) simplifies the denominator of this equation. Evaluating the total derivative with respect to τ gives

$$g_{\mu\nu}(x)\frac{\mathrm{d}u^\nu}{\mathrm{d}\tau} + \partial_\alpha g_{\mu\beta}(x)\, u^\alpha(\tau)\, u^\beta(\tau) = \frac{1}{2}\partial_\mu g_{\alpha\beta}(x)\, u^\alpha(\tau)\, u^\beta(\tau) \,, \tag{iv}$$

$$g_{\mu\nu}(x)\frac{\mathrm{d}u^\nu}{\mathrm{d}\tau} + \frac{1}{2}\big[\partial_\alpha g_{\mu\beta}(x) + \partial_\beta g_{\mu\alpha}(x) - \partial_\mu g_{\alpha\beta}(x)\big]\, u^\alpha(\tau)\, u^\beta(\tau) = 0 \,. \tag{v}$$

In the last step dummy indices have been relabelled. Multiplying through by the inverse of $g_{\mu\nu}(x)$ gives the result in equation (2.5).

The geodesics equations look a bit messy when written in this form. If you are familiar with the concept of a covariant derivative then they can be written in the more elegant form,

$$u^\alpha \nabla_\alpha u^\mu = 0 \,. \tag{vi}$$

2.2 The Geodesic Deviation Equation

A single freely falling test particle does not feel the effects of gravitational acceleration (a consequence of the equivalence principle). Therefore, a single particle cannot possibly function as a GW detector. A GW detector must consist of at least two freely falling particles, or an extended object of finite size. Therefore, our next task will be to obtain the equation governing how the separation of two freely falling particles changes with time.

Consider the first test particle, this follows a geodesic $x^\mu(\tau)$ which satisfies the geodesics equation in equation (2.5). Now consider a second, nearby test particle which follows a different geodesic $x^\mu(\tau)+\xi^\mu(\tau)$ which must also satisfy a version of the geodesic equations. Both geodesics are parameterised by their own proper times and the quantity $\xi^\mu(\tau)$ joins points with equal values of τ. The geodesic equations for the second particle are

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\big(x^\mu+\xi^\mu\big)+\Gamma^\mu{}_{\nu\rho}(x+\xi)\frac{\mathrm{d}}{\mathrm{d}\tau}\big(x^\mu+\xi^\mu\big)\frac{\mathrm{d}}{\mathrm{d}\tau}\big(x^\nu+\xi^\nu\big)=0\,. \tag{2.8}$$

Since we are dealing with two geodesics that are nearby we can assume that $\xi^\mu(\tau)$ is small[1]. Therefore, we can Taylor expand $\Gamma(x+\xi)=\Gamma(x+\xi)+\xi^\alpha\partial_\alpha\Gamma(x+\xi)+\mathcal{O}(\xi^2)$. Subtracting equation (2.5) from (2.8) gives the *equation of geodesic deviation*;

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\xi^\mu+2u^\nu\xi^\rho\Gamma^\mu{}_{\nu\rho}(x)+\xi^\alpha u^\nu u^\rho\partial_\alpha\Gamma^\mu{}_{\nu\rho}(x)=0\,. \tag{2.9}$$

The setup is illustrated in figure (2.2) and the equation is discussed further in box 2.2.

Box 2.2: Curvature and geodesic deviation

Again, the way we have written the equation of geodesic deviation appears a little messy. If you are familiar with the concepts of covariant differentiation and Riemannian curvature then they can be written in the more elegant form,

$$u^\alpha\nabla_\alpha\big(u^\beta\nabla_\beta\xi^\mu\big)=R^\mu{}_{\alpha\beta\gamma}u^\alpha u^\beta\xi^\gamma\,. \tag{i}$$

This is an important result in general relativity because it shows mathematically that it is the spacetime curvature which is responsible for the *tidal* gravitational forces which cause the divergences or convergences in the trajectories of nearby particles.

[1] Specifically, we assume that the separation of the particles is much smaller than the typical length scale over which the gravitational field changes.

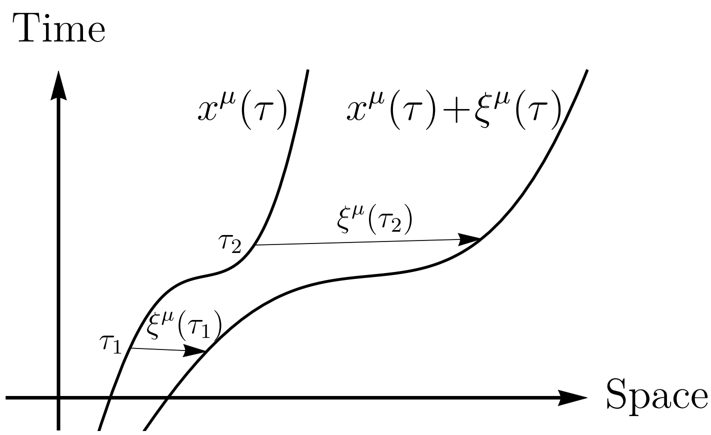

Figure 2.2: The deviation of two geodesics in spacetime. The vector $\xi^\mu(\tau)$ joins points on the two geodesics with equal values of the proper time parameter.

2.3 The Effect of GWs on Freely Falling Particles

The first part of this course described weak gravitational fields as small perturbations away from flat Minkowski space. Working in (nearly) Cartesian coordinates $x^\mu = \{ct, x, y, z\}$ the metric is given by $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$. We also saw that GWs admit an especially simple description in the transverse-traceless (TT) gauge. Recall that in the TT gauge a linearly polarised, plane-fronted, monochromatic GW travelling in the z-direction can be written as a combination of two linearly independent polarisation states:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ and } h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \exp\left[\mathrm{i}(\omega t - kz)\right]. \tag{2.10}$$

The GW angular frequency is ω and the GW travels at the speed of light, so $k = \omega/c$. The constants h_+ and $h_\times$ are the amplitudes of the two GW polarisation states and are assumed to be small ($h_{+,\times} \ll 1$). (The metric components are all real numbers; in equation (2.10) and henceforth it is to be understood that the real part is taken implicitly.) With this form of the metric the distance element becomes

$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= -c^2\mathrm{d}t^2 + \mathrm{d}z^2 + \Big\{1 + h_+ \exp\left[\mathrm{i}(\omega t - kz)\right]\Big\}\mathrm{d}x^2 \\ &\quad + \Big\{1 - h_+ \exp\left[\mathrm{i}(\omega t - kz)\right]\Big\}\mathrm{d}y^2 + 2h_\times \exp\left[\mathrm{i}(\omega t - kz)\right]\mathrm{d}x\mathrm{d}y\,. \end{aligned} \tag{2.11}$$

Given this metric, we can now solve the geodesics equations to see how test particles will move in this metric. The 4 spacetime coordinates of the worldline of a particle are

$$x^\mu(\tau) = \left\{ct(\tau),\, x(\tau),\, y(\tau),\, z(\tau)\right\}. \tag{2.12}$$

The geodesics equations are four, second order differential equations; therefore, we need to specify the initial spacetime position (4 constraints) and the initial four-velocity ($4+4=8$ constraints) to uniquely solve these equations. We will choose as our initial conditions a particle at rest (i.e. not moving in space) at an arbitrary point in space.

$$\begin{aligned} x(0) &= x_0\,, & y(0) &= y_0\,, & z(0) &= z_0\,, \\ \left.\frac{\mathrm{d}x}{\mathrm{d}\tau}\right|_{\tau=0} &= 0\,, & \left.\frac{\mathrm{d}y}{\mathrm{d}\tau}\right|_{\tau=0} &= 0\,, & \left.\frac{\mathrm{d}z}{\mathrm{d}\tau}\right|_{\tau=0} &= 0\,. \end{aligned} \tag{2.13}$$

We will also initialise our time coordinate such that $t(0) = 0$ But given these choices we are now forced to have our test particle "moving" forwards in time, $\mathrm{d}t/\mathrm{d}\tau\big|_{\tau=0} = c$, in order to satisfy the constraint in equation (2.3).

Having chosen the particle's position and velocity the geodesic equations allow us to calculate the particle acceleration. The $\mu = 1, 2, 3$ components of equation (2.5) give

$$\left.\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2}\right|_{\tau=0} = \left[\Gamma^x{}_{tt}(x)\right]_{\tau=0}, \tag{2.14}$$

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}\tau^2}\right|_{\tau=0} = \left[\Gamma^y{}_{tt}(x)\right]_{\tau=0}, \tag{2.15}$$

$$\left.\frac{\mathrm{d}^2z}{\mathrm{d}\tau^2}\right|_{\tau=0} = \left[\Gamma^z{}_{tt}(x)\right]_{\tau=0}. \tag{2.16}$$

Therefore, we need to compute the $\Gamma^x{}_{tt}$, $\Gamma^y{}_{tt}$, and $\Gamma^z{}_{tt}$ *connection coefficients*. This can be done using the definition in equation (2.7) and the metric in equation (2.10). It is a straightford exercise in differentiation to compute these connection coefficients, Taylor expand them to linear order in h_+ and $h_\times$, and finally to show that

$$\Gamma^x{}_{tt}(x) = \Gamma^y{}_{tt}(x) = \Gamma^z{}_{tt}(x) = 0\,. \tag{2.17}$$

Just these three components are enough to show that all the spatial components of the acceleration in equation (2.14) are zero. Freely falling particles that start at rest in our coordinates remain at rest. The worldline of our particle must therefore be

$$t(\tau) = \tau\,, \quad x(\tau) = x_0\,, \quad y(\tau) = y_0\,, \quad z(\tau) = z_0\,. \tag{2.18}$$

Don't misunderstand this result as implying that GWs have no effect on free particles. All this result is saying is that the particles remain at rest in our chosen system of coordinates. To understand the true effect of the GW we need to calculate a geometrical quantity (i.e. a scalar) with a value independent of the coordinates used to describe it.

One way to do this is to place many test particles in a ring with around an observer centred at the origin. As we have seen, the spatial coordinates of the particles and the observer don't change with time. We choose the coordinates of the particles such that

$$z_0 = 0\,, \quad x_0^2 + y_0^2 = R_0^2\,. \tag{2.19}$$

Now compute the proper distance from the origin to the ring of test particles. Although the ring appears to be circular in our coordinate, this distance will nevertheless depend on time and also on which direction we choose. It is natural to introduce 2 dimensional plane polar coordinates (r, ϕ) to describe the plane containing the ring;

$$x_0 = r\cos\phi\,, \quad y_0 = r\sin\phi\,. \tag{2.20}$$

Using equation (2.11), the proper distance to the ring is given by (see box 2.3)

$$R(\phi, t)/R_0 = 1 + \frac{h_+}{2}\exp(\mathrm{i}\omega t)(\cos^2\phi - \sin^2\phi) + h_\times \exp(\mathrm{i}\omega t)\cos\phi\sin\phi\,. \tag{2.21}$$

The effect of the GW on the proper distance to the ring is illustrated in figure (2.3).

Box 2.3: The proper distance

The proper distance from the origin to the ring of test particles is obtained by integrating the distance element in equation (2.11),

$$R(\phi, t) = \int_{\text{Origin}}^{\text{Ring}} \mathrm{d}s. \tag{i}$$

Integrating along a curve with $\mathrm{d}t = \mathrm{d}z = \mathrm{d}\phi = 0$ gives

$$R(\phi, t) = \int_0^{R_0} \mathrm{d}r \left(\left\{ 1 + h_+ \exp\left[\mathrm{i}(\omega t - kz)\right] \right\} \left(\frac{\mathrm{d}x}{\mathrm{d}r}\right)^2 + \right. \tag{ii}$$
$$\left. \left\{ 1 - h_+ \exp\left[\mathrm{i}(\omega t - kz)\right] \right\} \left(\frac{\mathrm{d}y}{\mathrm{d}r}\right)^2 + 2h_\times \exp\left[\mathrm{i}(\omega t - kz)\right] \frac{\mathrm{d}x}{\mathrm{d}r}\frac{\mathrm{d}y}{\mathrm{d}r} \right)^{1/2}.$$

Using $\mathrm{d}x/\mathrm{d}r = \cos\phi$ and $\mathrm{d}y/\mathrm{d}r = \sin\phi$ from equation (2.20), and expanding to linear order in the GW amplitude (h_+ and $h_\times$) we obtain equation (2.21).

Exercise 2.1:
Calculate how the *proper area* inside the ring changes over time.

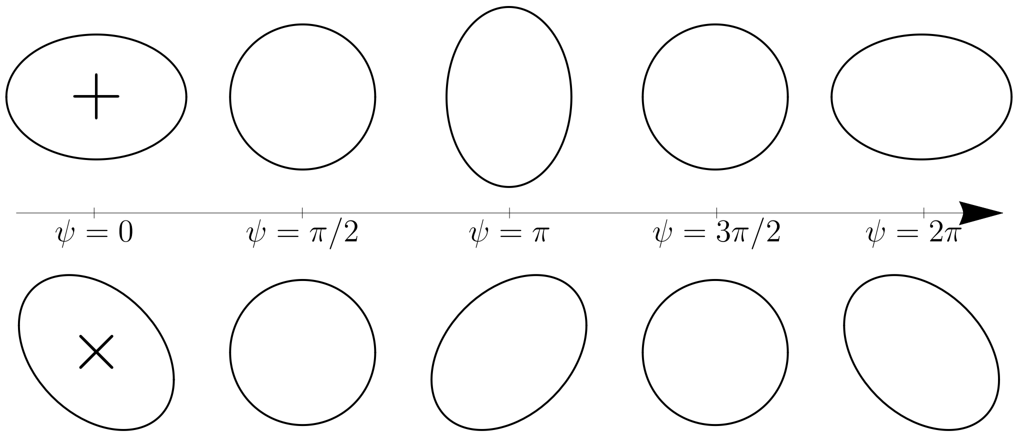

Figure 2.3: The effect of a GW on the proper distance from the origin to an initially circular ring of freely falling test particles. Remember, in our coordinates the particles are not moving, what is plotted here is the *proper distance* $R(\phi, t)$ in equation (2.21). The two rows show the effects of the two linearly independent GW polarisations: the top row shows a *plus polarised* GW with $h_+ \neq 0$ and $h_\times = 0$, and the bottom row shows a *cross polarised* GW with $h_\times \neq 0$ and $h_+ = 0$. Time increases from left to right and five snapshots of the proper distances are shown at equally spaced intervals of the GW phase $\psi = \omega t$.

2.3.1 Laser Interferometry

One practical way of measuring this effect is to use light. If the ring of test particles are replaced by reflecting mirrors, and the observer at the origin emits a circularly symmetric pulse of light at time $t = 0$ then the light will bounce back to the origin at some later time. In the absence of any GWs (i.e. $h_+ = h_\times = 0$) then all of the light will return to the origin after a time R_0/c (this is the proper time as measured by a clock carried by the observer at the origin). If instead there is, say, a plus polarised GW passing through the experiment (i.e. $h_\times = 0$ but $h_+ \neq 0$) then the light emitted in different directions will return after different amounts of time. Measuring these time differences is one way to detect GWs.

In practice, a experimental setup similar to that described above can be achieved using *laser interferometry.* Instead of a ring of test particles, two mirrors are placed at the end of kilometer scale pipes. The mirrors are suspended from the ceiling; they are not really freely falling test particles, but they do respond as such to motion in the horizontal directions on timescales less than the suspension oscillation period. Instead a pulse of light, a continuous laser is split and directed to both mirrors. The light is reflected back to the centre and recombined onto a photodetector where the two beams interfere coherently. Extremely small changes in distance, e.g. due to a passing GW, can be measured by looking at the changing interference pattern. Several collaborations around the world have constructed laser interferometers with the aim of observing GWs. Currently, the most sensitive of these are the two LIGO observatories [see figure (2.4)].

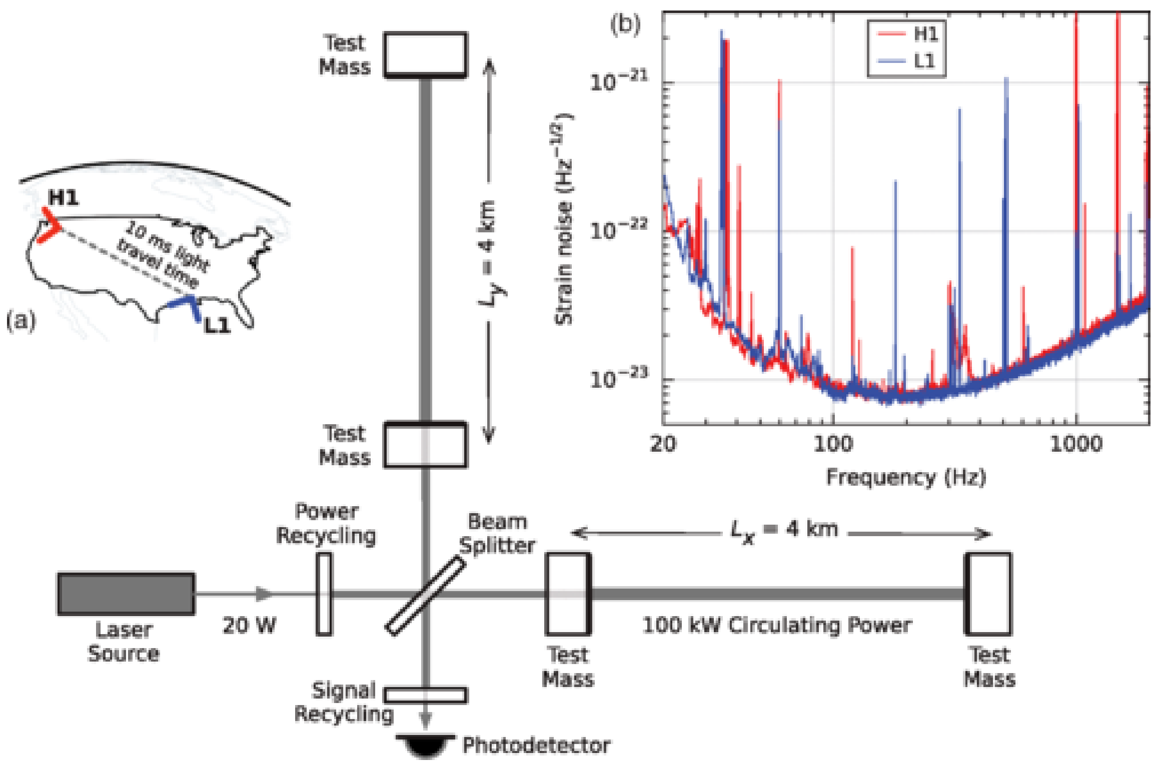

Figure 2.4: There are two LIGO detectors, their locations within the USA are shown in the (a) inset. The two 4 km detectors are built to the same basic design and have achieved unprecedented strain sensitivities; the (b) inset shows that at frequencies $f \approx 100\,\text{Hz}$ the detectors can measure strains (strain $\equiv$ length change / length) of less than one part in 10^{23}. The main figure shows a simplified diagram of one of the advanced LIGO interferometers. Figure reproduced from B. P. Abbott *et al.* (LIGO Scientific Collaboration and Virgo Collaboration) Physical Review Letters **116**, 061102 - Published 11 February 2016.

2.4 The Effect of GWs on Connected Particles

The coordinates used above proved extremely convenient for describing freely falling particles. But many situations involve particles which interact in various complicated ways.

For example, *rigid bodies* consist of a collection of particles joined together by rigid struts.

Let us consider a simpler system of interacting particles; two point particles of mass m connected by a massless spring with spring constant k, damping constant ν, and an unstretched length l_0 [see figure (2.5)].

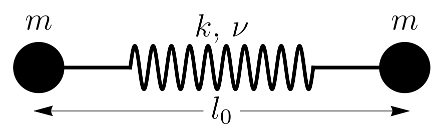

Figure 2.5: Two particles with mass m are joined by a spring with stiffness constant k, damping constant ν, and an natural length l_0.

For simplicity, we will align the system with the x-axis so that we only have to deal with motion in one dimension (i.e. we will always assume $y = z = 0$). Let $x_1(t)$ and $x_2(t)$ denote the position of the two particles along the x-axis as a function of the coordinate time. In flat space (i.e. no GWs; the metric is simply $g_{\mu\nu}(x) = \eta_{\mu\nu}$) the stretched length of the spring is given by

$$l(t) = \int_{x_1(t)}^{x_2(t)} \mathrm{d}x = x_2(t) - x_1(t)\,. \tag{2.1}$$

We will assume that all the motions of the system are much smaller than the speed of light so that we can neglect any special relativistic effects. The Newtonian equations of motions for the two particle system are obtained by considering the spring force and the damping force acting on the two masses:

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big[x_1(t)\big] = -k\big[l_0 - l(t)\big] - \nu\frac{\mathrm{d}}{\mathrm{d}t}\big[l_0 - l(t)\big]\,, \tag{2.2}$$

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big[x_2(t)\big] = -k\big[l(t) - l_0\big] - \nu\frac{\mathrm{d}}{\mathrm{d}t}\big[l(t) - l_0\big]\,. \tag{2.3}$$

This is a coupled system of differential equations. It can be simplified by defining a new variable corresponding to the *displacement* $\xi = l(t) - l_0$. If we also define $\omega_0^2 = 2k/m$ and $\gamma = \nu/m$, then we obtain the familiar equation for a damped harmonic oscillator,

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} + 2\gamma\frac{\mathrm{d}\xi}{\mathrm{d}t} + \omega_0^2\xi = 0\,. \tag{2.4}$$

Now we consider what happens when a GW comes past. The Newtonian equations of motion derived above are still valid (assuming that GW is weak and that all of the velocities are non relativistic). The only important difference is now the spring will exert a force proportional to the change in its *proper length*, not just its coordinate length. Just as in box 2.3, the proper length is given by an integral along the spring of the metric line interval. So we must replace the length in equation (2.1) with

$$l(t) = \int_{\text{Particle 1}}^{\text{Particle 2}} \mathrm{d}s\,. \tag{2.5}$$

Because we are integrating along the spring we can use the distance line element in equation (2.11) with $\mathrm{d}t = \mathrm{d}y = \mathrm{d}z = 0$;

$$l(t) = \int_{x_1(t)}^{x_2(t)} \mathrm{d}x\,\sqrt{1 + h_+ \exp(\mathrm{i}\omega t)} = \big[x_2(t) - x_1(t)\big]\left[1 + \frac{1}{2}h_+ \exp(\mathrm{i}\omega t) + \mathcal{O}(h_+^2)\right] \tag{2.6}$$

$$= x_2(t) - x_1(t) + \frac{l_0}{2}h_+ \exp(\mathrm{i}\omega t) + \mathcal{O}(h_+^2)\,. \tag{2.7}$$

In the last step we have used the fact that the factor $\big[x_2(t) - x_1(t)\big]$ multiplying h_+ can be replaced by l_0 at the required order.

Exercise 2.2:
By substituting the proper length of the spring in equation (2.7) into the system of equations (2.2) and (2.3) show that the displacement $\xi(t) = l(t) - l_0$ now satisfies

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} + 2\gamma\frac{\mathrm{d}\xi}{\mathrm{d}t} + \omega_0^2\xi = \frac{-l_0\omega^2}{2}h_+ \exp(\mathrm{i}\omega t)\,. \tag{i}$$

This is the equation of motion for a damped, *driven* oscillator.

The driving force for the oscillator equation derived in exercise 2.2 is provided by the GW. Therefore, we can think of the GW as a acting on our system just like a *Newtonian force*; here, the magnitude of the force is $F = ml_0\omega^2 h_+ \exp(\mathrm{i}\omega t)/2$.

In this example we choose to use a spring of natural length l_0 aligned along the x-axis. This was an arbitrary choice, and we could have chosen to put one of the masses at the origin and the second mass anywhere in the $x - y$ plane. If we did this then the GW force in the spring some general point in the plane, $\xi^i = (x,\, y,\, z = 0)$, would be given by

$$F_i = \frac{m}{2}\frac{\mathrm{d}^2 h_{ij}}{\mathrm{d}t^2}\xi^j = -\frac{1}{2}m\omega^2\begin{pmatrix} yh_\times + xh_+ \\ xh_\times - yh_+ \\ 0 \end{pmatrix}\exp(\mathrm{i}\omega t)\,, \tag{2.1}$$

where $h_{ij}(x)$ is the TT gauge metric perturbation. The spatial pattern of this force is illustrated in figure (2.6). Figures (2.3) and (2.6) are the two common ways of visualising the effects of a GW. There is another way to derive the result in equation (2.1) which uses the geodesic deviation equation. This helps to make it clear that is the tidal gravitational forces, described by the Reimann tensor, which are responsible for driving the oscillator.

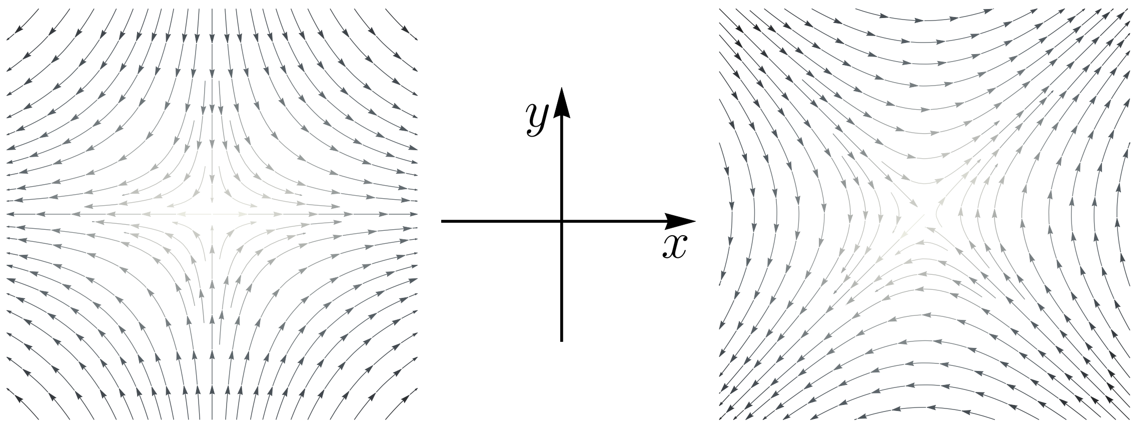

Figure 2.6: The pattern of forces, F_i in equation (2.1), in the xy plane from a GW in the detector frame. The two panels show the effects of the two linearly independent GW polarisations: plus (+, left) and cross (×, right). There shading of the arrows indicates the magnitude of the force, which increases further from the origin.

2.4.1 Resonant Bar Detectors

We have analysed the response of two masses connected by a spring to an incident GW. This is a simple model for a *resonant bar detector* These instruments are historically important because they were the first significant experimental effort to detect GWs. These early efforts, spearheaded by Joseph Weber [see figure (2.7)], consisted of $\sim 3\,\mathrm{m}$ cylindrical aluminium bars. As we have seen, when a GW passes over the bar it exerts a force between the two ends which acts to periodically stretch and compress the bar, exciting small mechanical oscillations along its length. A mechanical transducer attached to the end of the bar can be tuned to amplify small oscillations near a particular frequency and a piezoelectric crystal turns these mechanical oscillations into an electrical signal.

A small number of resonant detectors descended from Weber's bars continue to operate, however they have largely been superseded by the development of laser interferometers. However, many of the techniques pioneered by during these early efforts continue to prove useful. For example, Weber realised that a single detector would never be enough, regardless of how carefully it was constructed. Even if that single detector recorded a large signal it would be impossible to say with any certainty whether it was due to a GW or some nearby environmental effect. Therefore, Weber positioned a network of his resonant bars across the United States and looked for events that occured (almost) coincidently across the network. The same technique is used currently by the LIGO and Virgo detectors.

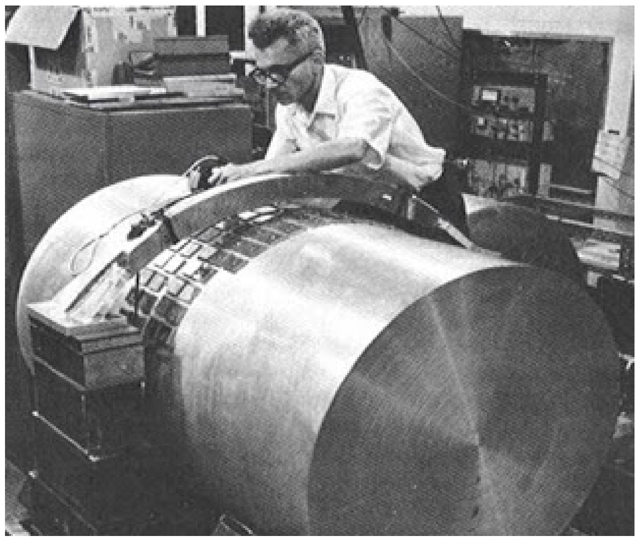

Figure 2.7: Joseph Weber working on a resonant bar detector at the University of Maryland c. 1965. Image reproduced from `https://umdrightnow.umd.edu/news/gravitational-wave-detection-validates-einstein-early-work-umd-physicists`.

2.5 Discussion

Although gravitational wave science has really only just begun it already has a long history. The existence of GWs was first predicted by Einstein c. 1915. There was initially some confusion over the whether these waves had any real physical effects. This confusion was due in a large part to the issues of gauge freedom discussed in the first part of this course, and it wasn't completely resolved until the 1960s.

The first serious experimental effort to detect gravitational waves was begun in the 1960s by Joseph Weber and used a network of resonant bar detectors. We have seen in the second part of this course how a GW exerts a force and excites mechanical oscillations in such an instrument. However, the first evidence for GWs came from another, indirect source. In the 1970s radio observations of the binary pulsar PSR B1913+16 by Hulse and Taylor showed that the system was losing energy at exactly the rate predicted by the quadrupole formula. Hulse and Taylor were awarded the Nobel prize for physics in 1993; the first Nobel prize awarded in the field of gravitational wave astronomy.

We have also seen in the second part of this course how laser interferometry can be used to measure small changes in the distance between test particles. The idea of using laser interferometry to detect gravitational waves was developed during the 1970s and 80s and construction began on the two LIGO detectors in the mid 1990s. After an uneventful early observation period in the mid 2000s, these efforts were rewarded with the first direct detection of gravitational waves in September 2015. These gravitational waves were produced during a collision of two distant black holes, each around 30 times the mass of our sun, which occurred almost 10^9 years ago [see figure (2.8)]. Such a system was described, to a first approximation, in the first part of the course. Three pioneers in the development of the field received the 2017 Nobel prize in physics; the second Nobel prize awarded in the field.

In addition to LIGO, there is a growing number of other ground-based laser interferometers hunting for gravitational waves: Virgo (in Italy), Kagra (in Japan), LIGO-India. There are also advanced plans for ESA to launch a space-based gravitational wave interferometer called LISA in the early 2030s.

Finally, there are other possibilities, besides resonant detectors and interferometers, for detecting gravitational waves that have not been discussed in this course. Such methods include the use radio telescopes to time networks of pulsars throughout the galaxy to look for small changes in the arrival timing of the radio pulses. These pulsar timing arrays (PTAs) will be able to detect very low frequency GWs generated by extremely heavy black holes.

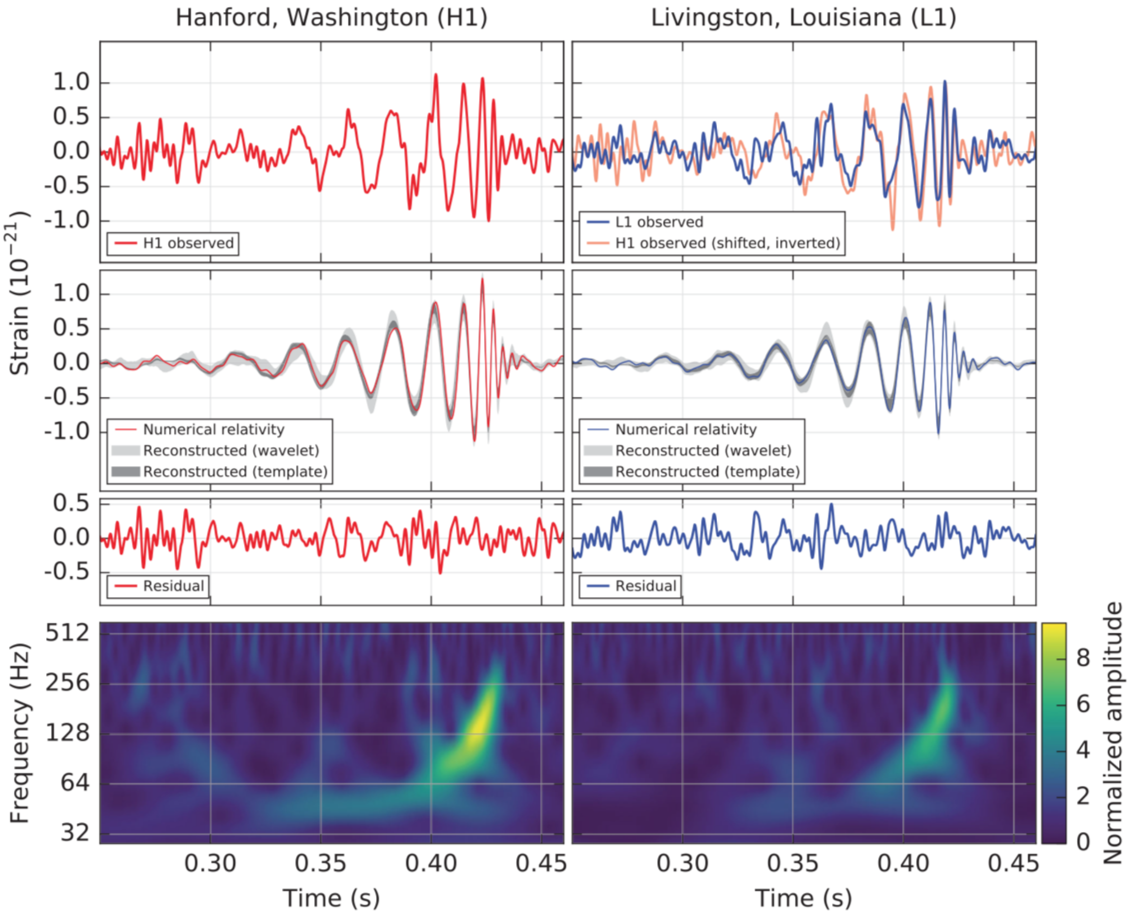

Figure 2.8: The first observation of GWs from the binary black hole system GW150914. The two columns show data from the two LIGO detectors in Washington (left) and Louisiana (right). The first row shows the measured GW strain (bandpass filtered between 35 Hz and 350 Hz). The second row shows results from a theoretical model of the signal (the same bandpass filter has been applied). The third row shows the residuals between the measured data and the theoretical model. Finally, the fourth row shows a time frequency "spectrogram" of the measured data showing how the frequency rapidly increases from ~ 30 Hz to ~ 150 Hz in a manner called a "chirp". Figure reproduced from B. P. Abbott *et al.* (LIGO Scientific Collaboration and Virgo Collaboration) Physical Review Letters **116**, 061102 - Published 11 February 2016. This paper announced the first detection of GWs. It is a beautifully written paper, which largely accessible to non-specialists, and it represents a milestone in the history of science. The paper is freely available online and I strongly encourage you to read it.