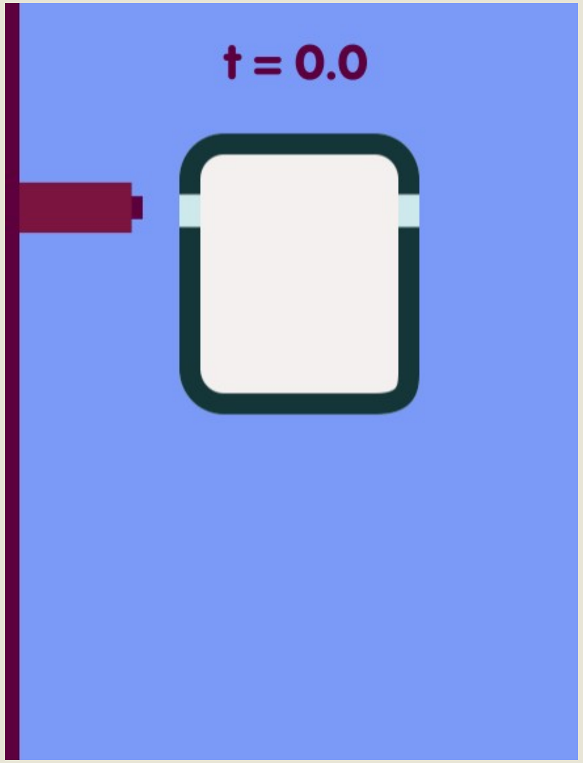
Acceleration in Special Relativity and Mach's Principle

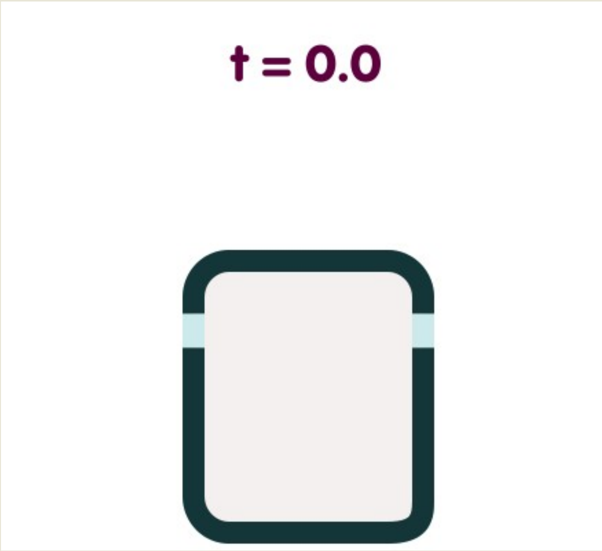
Edgar Gasperín

CENTRA
Instituto Superior Técnico
Universidad de Lisboa

Equivalence principle

Weak Equivalence Principle: all particles fall at the same rate in a gravitational field, independent of their mass and composition





The **equivalence principle**: In a freely falling (non-rotating) laboratory occupying a small region of spacetime, the laws of physics are those of special relativity.

Special Relativity

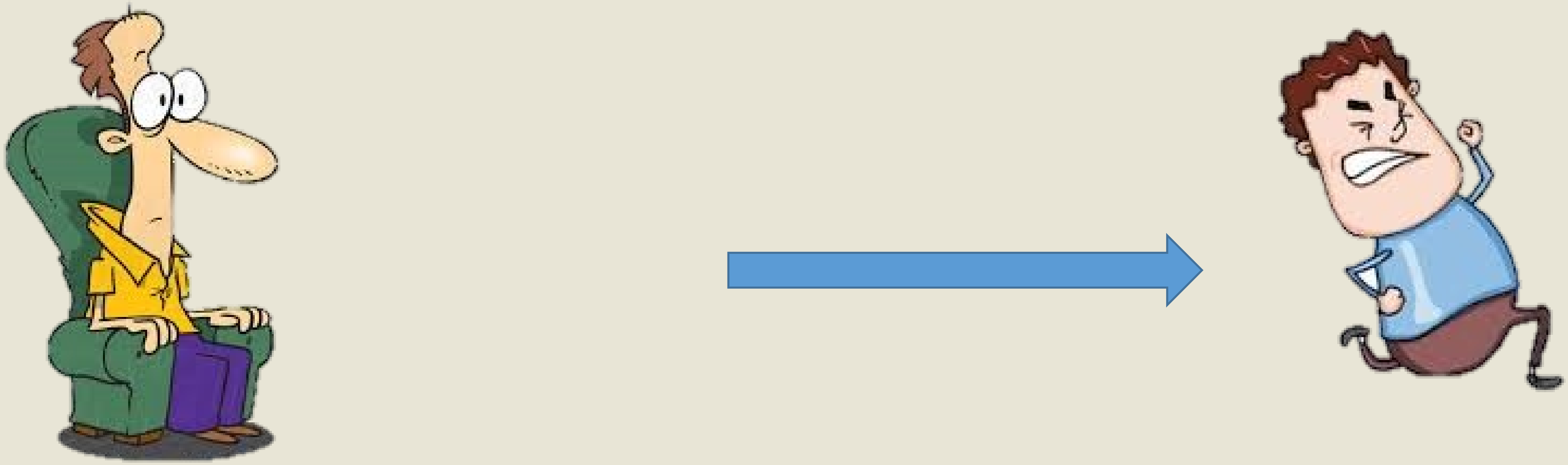
The **equivalence principle**: In a freely falling (non-rotating) laboratory occupying a small region of spacetime, the laws of physics are those of special relativity.

Special Relativity = Flat Lorentzian Geometry!

Objects moving in the absence of gravitational field

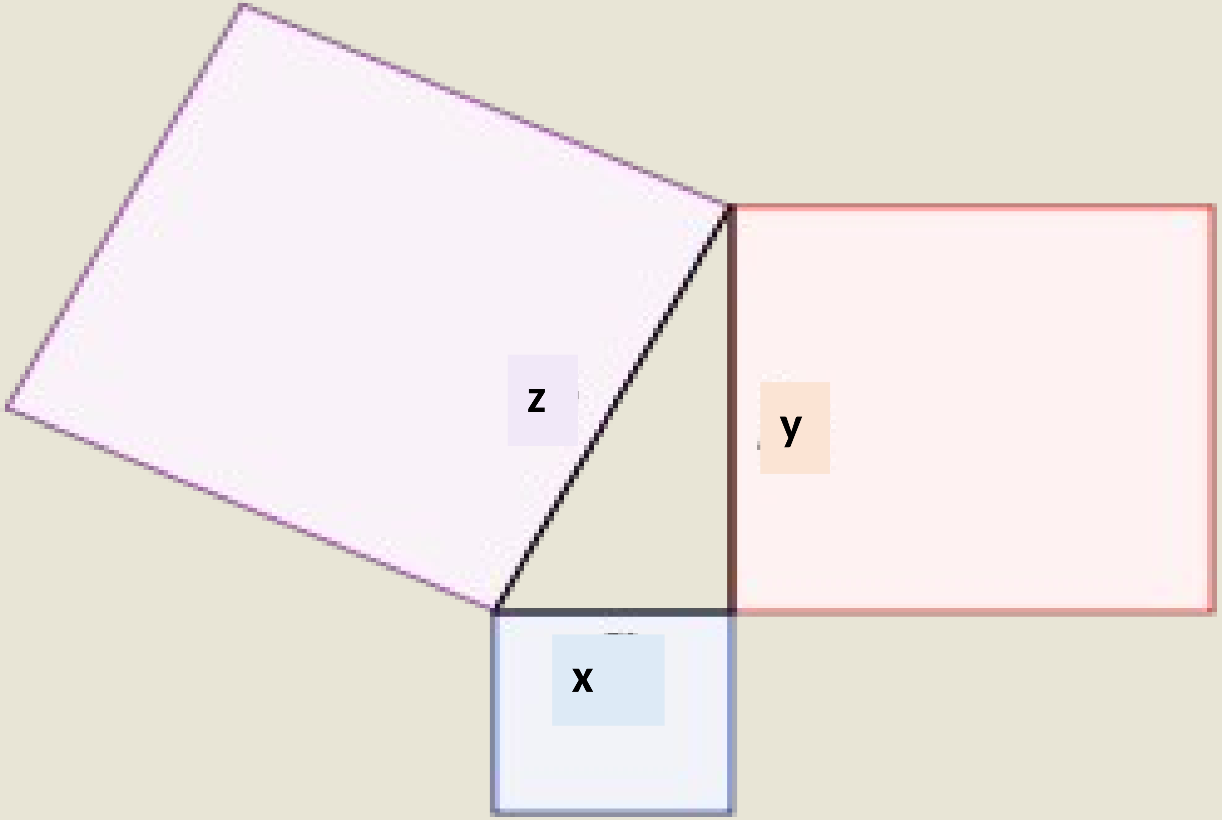
General Relativity = Curved Lorentzian Geometry!

Objects moving in the presence of gravitational field



Flat space (Riemannian) = Euclidean geometry

Pato Donald no país da MateMágica!

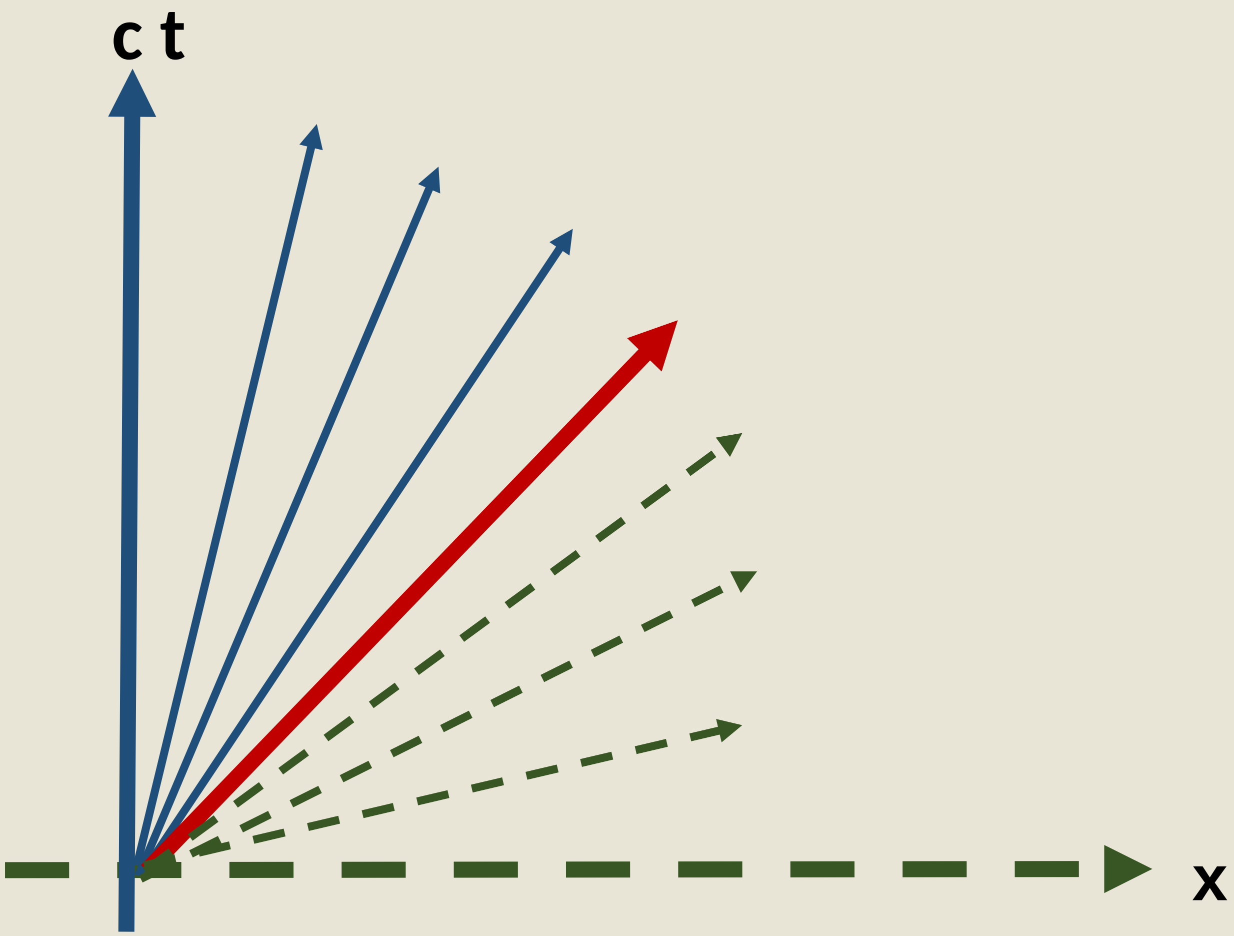


$$z^2 = x^2 + y^2$$

Metric $dz^2 = dx^2 + dy^2$

Flat space (Lorentzian) = Minkowski space-time

Metric $c^2 d\tau^2 = - c^2 dt^2 + dx^2$



- **Time**like vectors
- **Space**like vectors
- Null or **Light**like vectors

Galilean view vs Einstein view of Past and Future

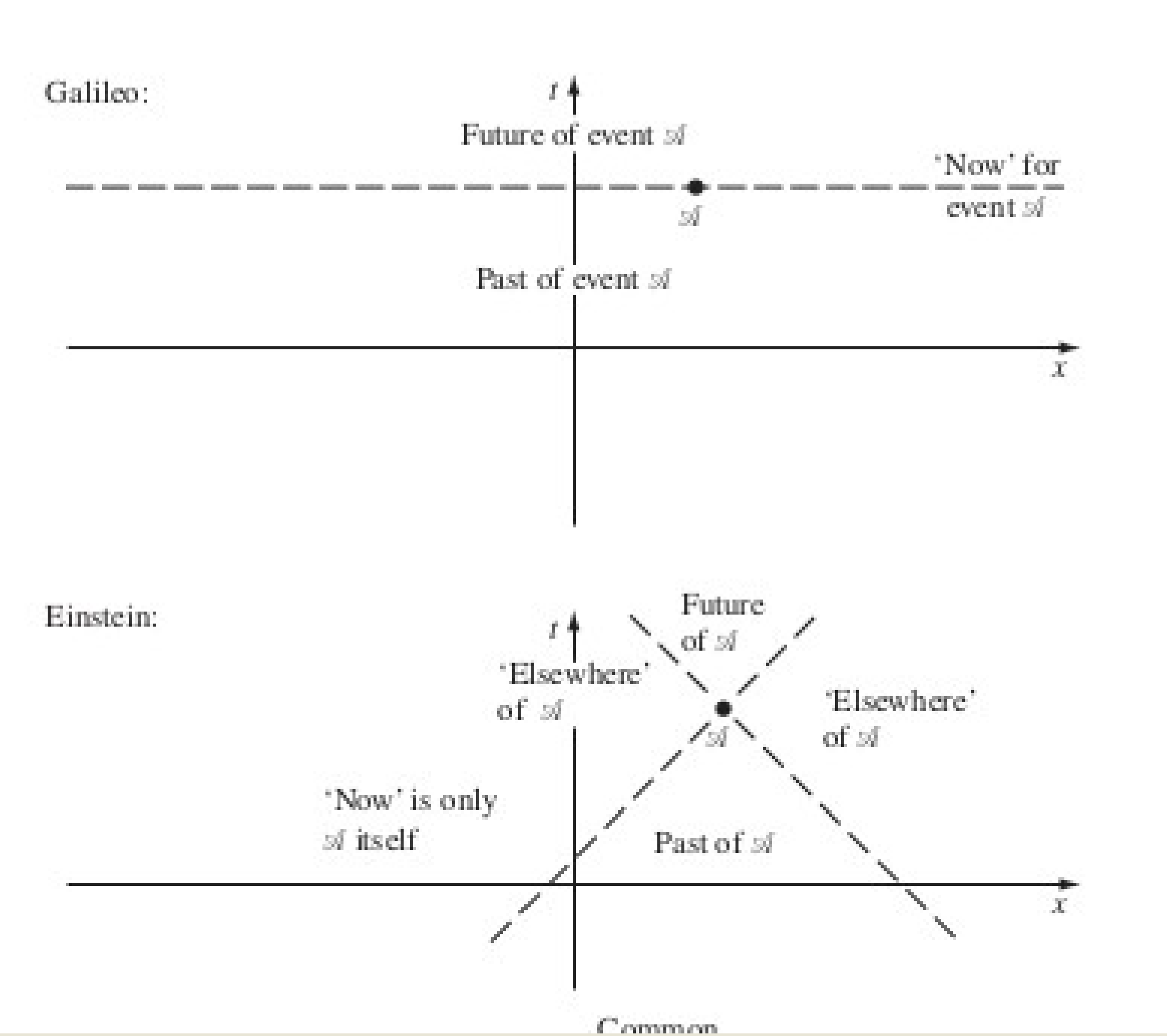


Image: General Relativity. B. Schutz

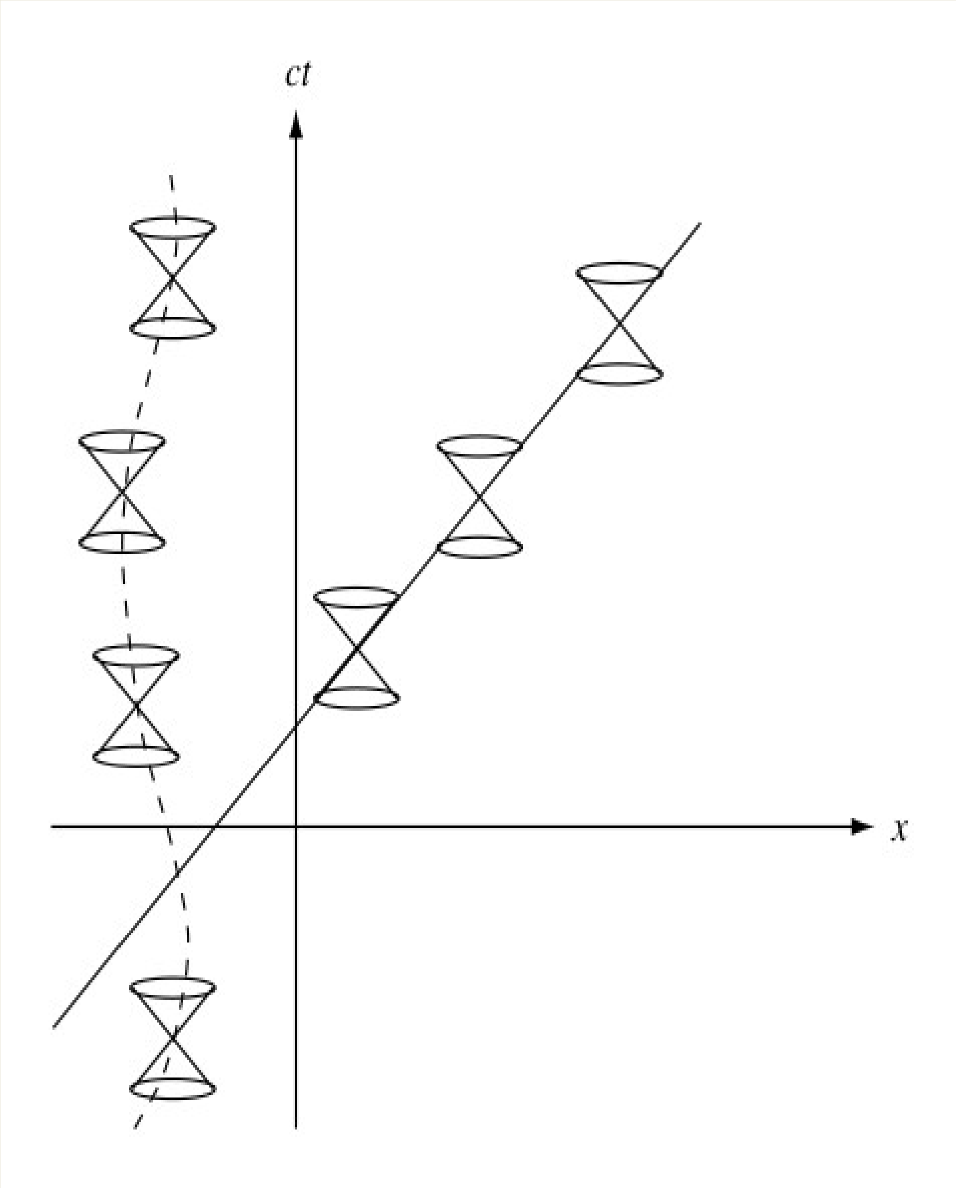
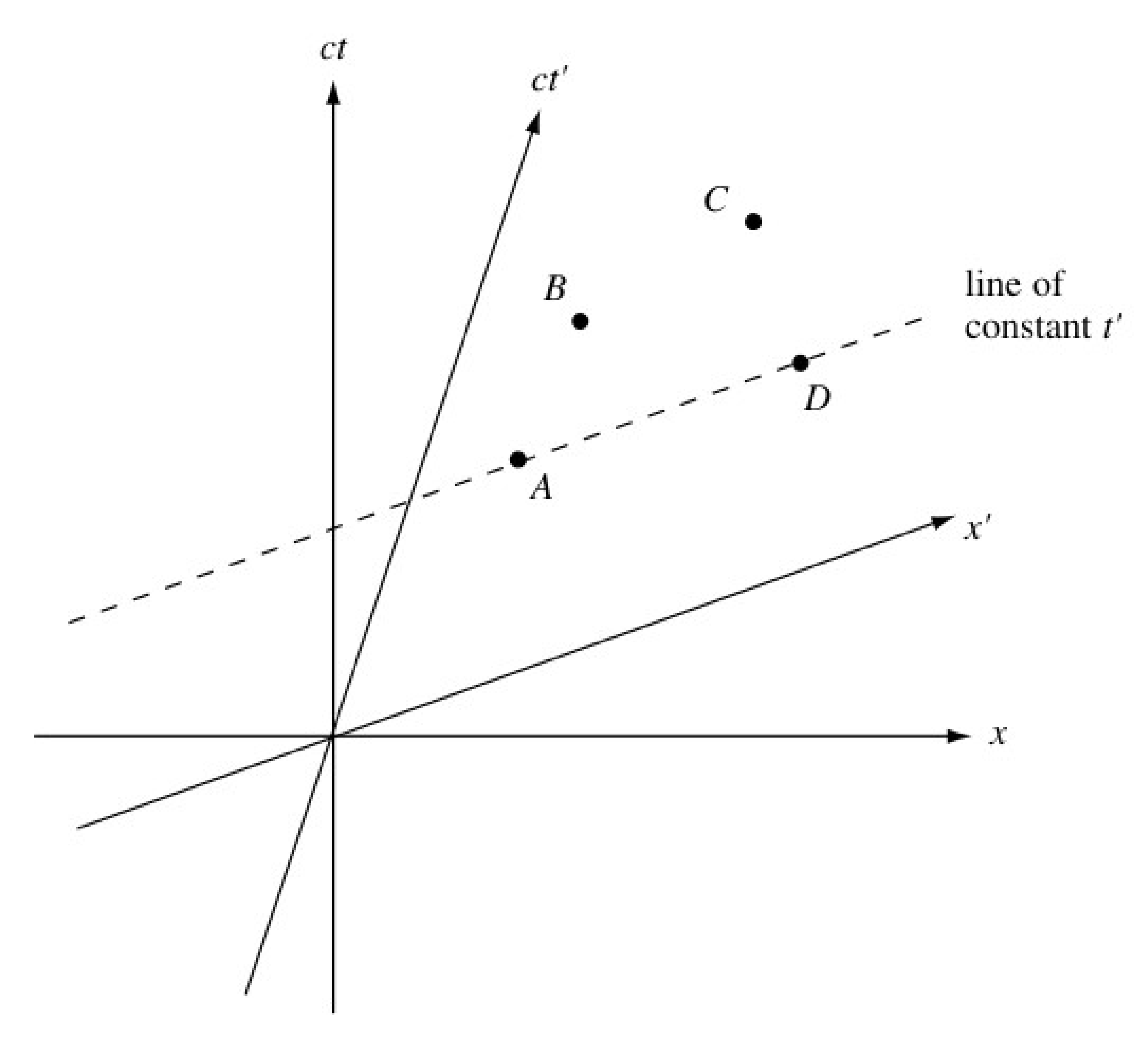
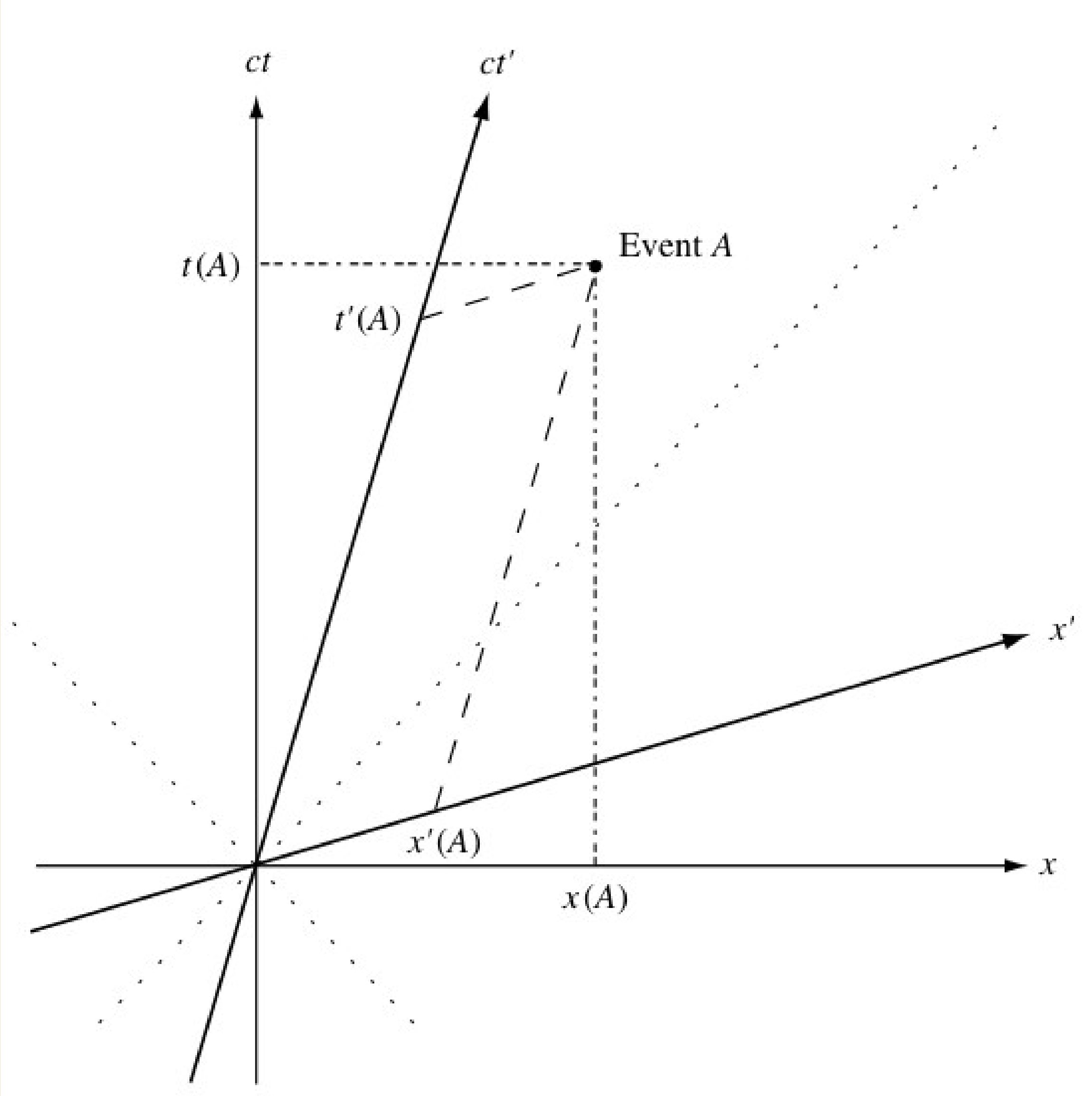


Image: General Relativity: An introduction for physicists. Hobson, Efstathiou & Lasenby

Simultaneity in Special Relativity

One can “read” the time dilation effect from the diagram without doing any computation





Images: General Relativity: An introduction for physicists. Hobson, Efstathiou & Lasenby

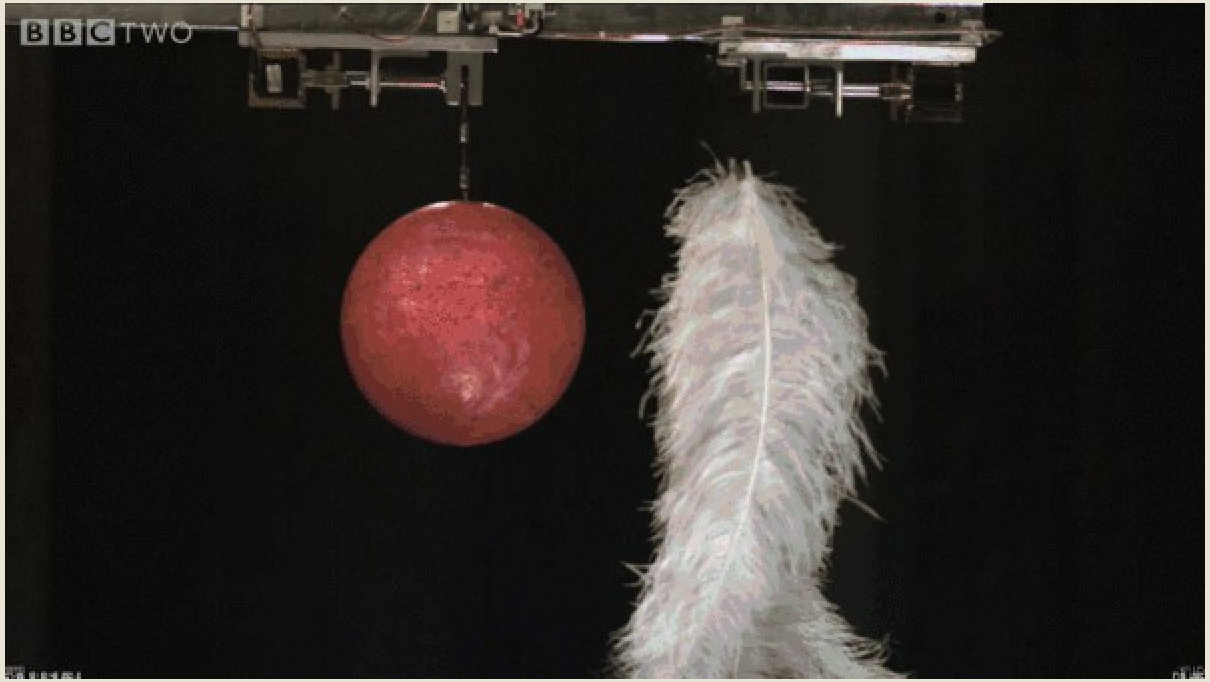
Curves of constant acceleration

Newtonian

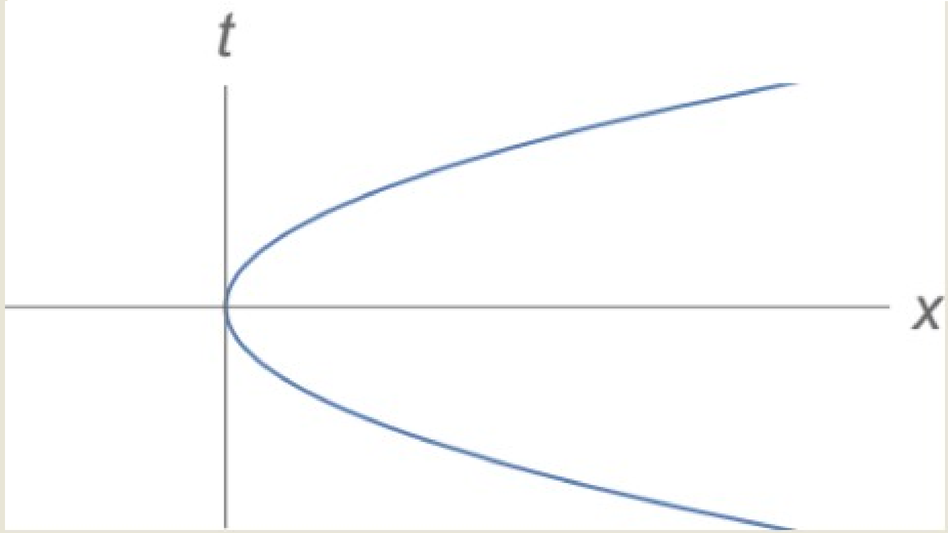
$$\frac{d^2x}{dt^2} = a$$
$$x(t) = \frac{a}{2}t^2 + v_0 t + x_0$$

Parametric curve

$$x(\lambda) = \frac{a}{2}\lambda^2$$
$$t(\lambda) = \lambda$$

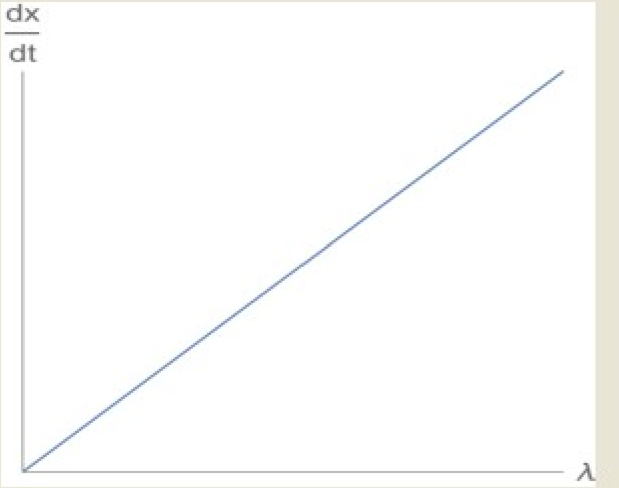


For simplicity take $x_0 = v_0 = 0$



Parabola in spt diagram

$$\frac{dx}{dt} = at$$



Curves of constant acceleration

Definitions

$$\boldsymbol{x} = (t(\lambda), x(\lambda))$$

Worldline

$$\dot{\boldsymbol{x}} = (dt/d\lambda, dx/d\lambda)$$

Tangent vector

$$c d\tau = \sqrt{-||\dot{\boldsymbol{x}}||^2_{Mink}}$$

Proper time

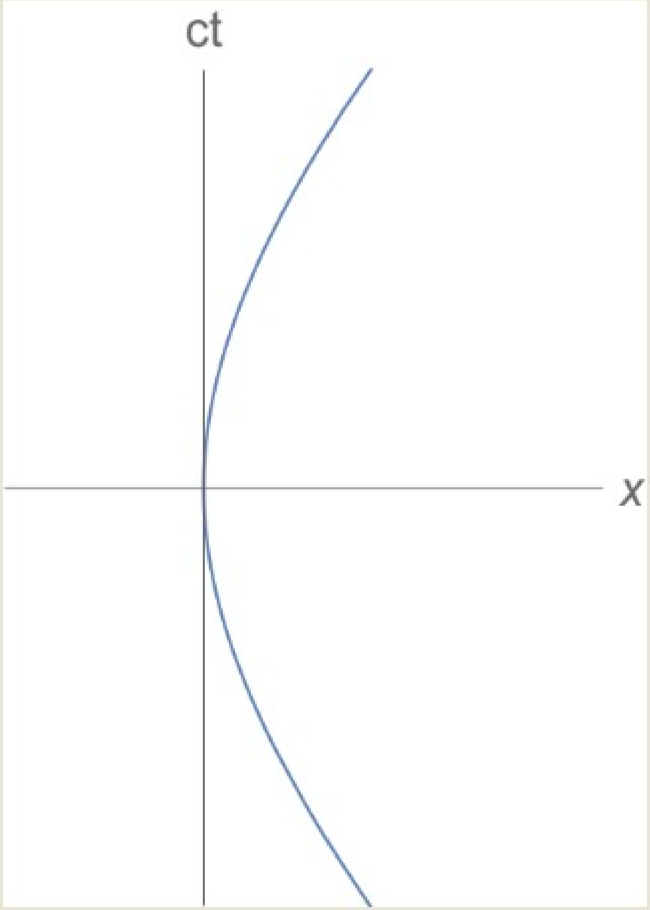
$$\boldsymbol{u} = \frac{1}{c}\left(\frac{dt}{d\tau}, \frac{dx}{d\tau}\right)$$

Proper 4-velocity

$$\boldsymbol{a} = \frac{1}{c^2}\left(\frac{d^2t}{d\tau^2}, \frac{d^2x}{d\tau^2}\right)$$

Proper 4-acceleration

Spacetime diagram



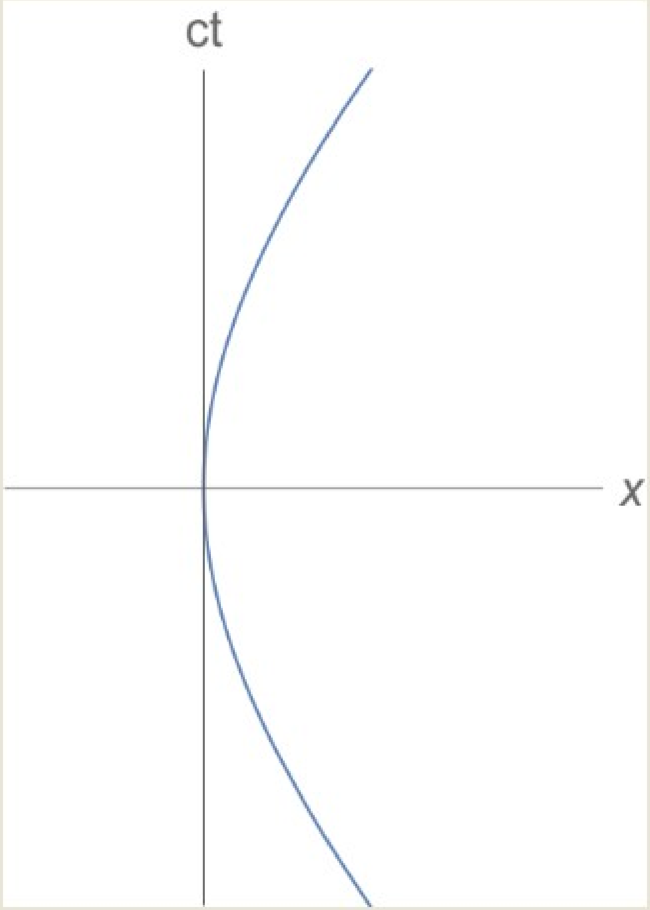
Hyperbola in spt

$$x(\lambda) = \frac{c^2}{a_p}\left(\cosh\left[\frac{a_p}{c}\lambda\right] - 1\right)$$

$$t(\lambda) = \frac{c}{a_p}\sinh\left[\frac{a_p}{c}\lambda\right]$$

Curves of constant acceleration

Special Relativity

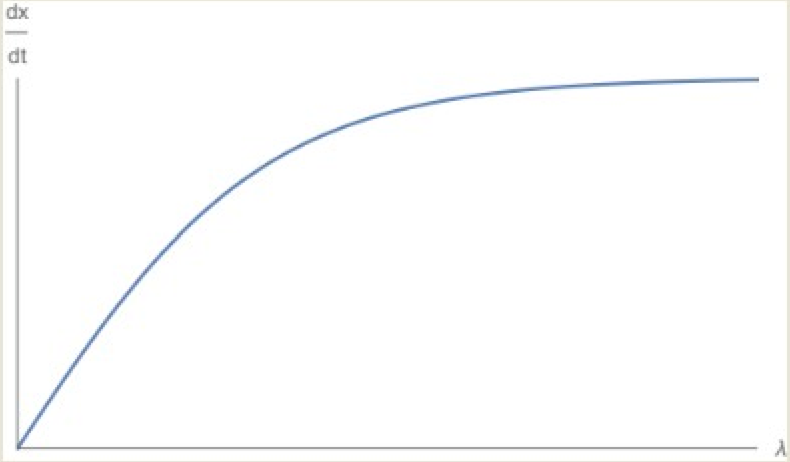


Hyperbola in spt diagram

$$\dot{\boldsymbol{x}} = \left(\cosh\left[\frac{a_p}{c}\lambda\right],\ c\sinh\left[\frac{a_p}{c}\lambda\right]\right)$$

$$||\dot{\boldsymbol{x}}||^2_{Mink} = -c^2 \implies \tau = \lambda$$

$$\frac{dx/d\lambda}{dt/d\lambda} = \frac{dx}{dt} = c\tanh\left[\frac{a_p}{c}\lambda\right],$$



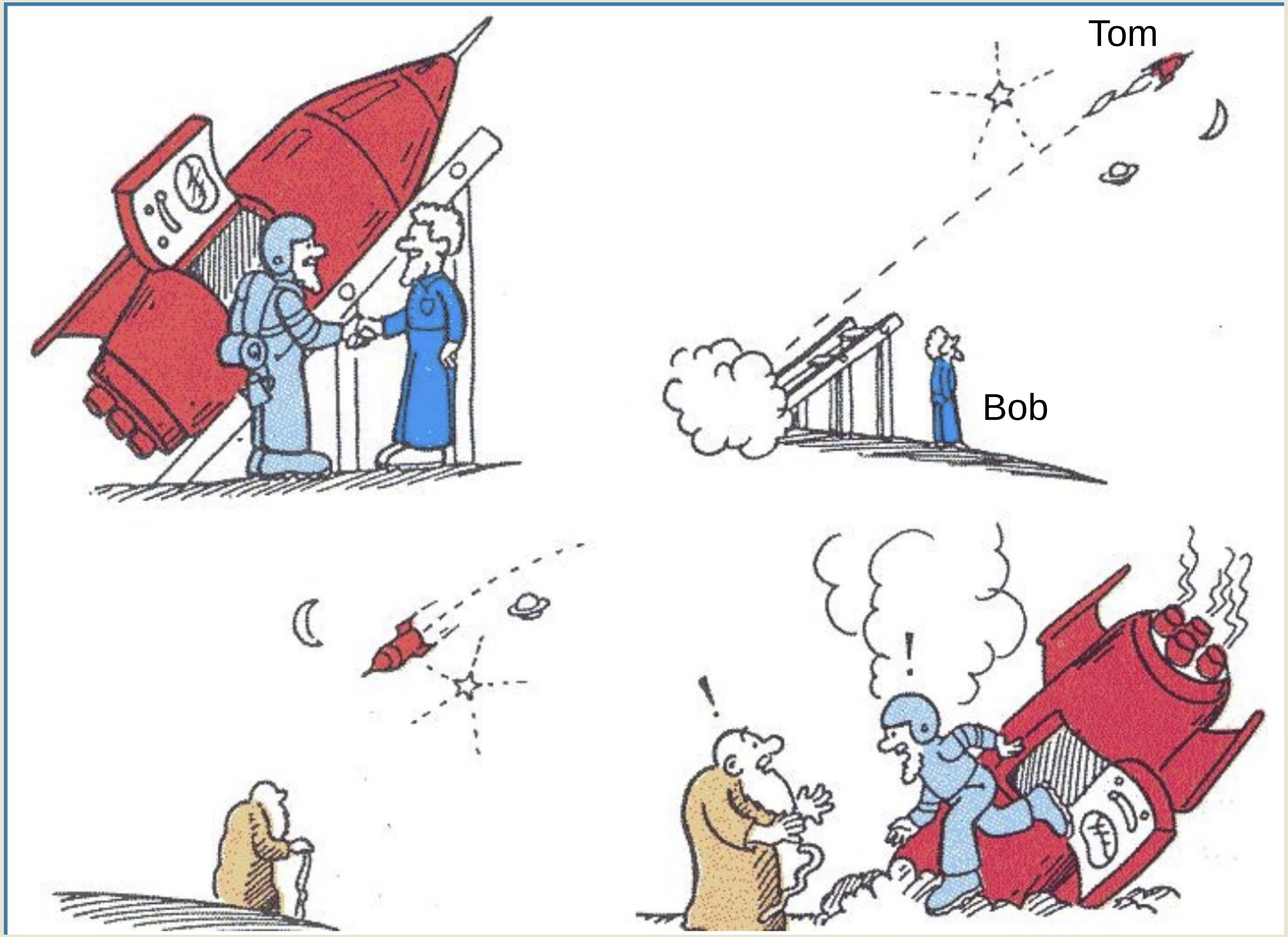
$$\boldsymbol{u} = \left(\frac{1}{c}\cosh\left[\frac{a_p}{c}\tau\right],\ \sinh\left[\frac{a_p}{c}\tau\right]\right)$$

$$\boldsymbol{a} = \frac{1}{c}\left(\frac{a_p}{c^2}\sinh\left[\frac{a_p}{c}\lambda\right],\ \frac{a_p}{c}\cosh\left[\frac{a_p}{c}\lambda\right]\right)$$

$$||\boldsymbol{a}||^2_{Mink} = \frac{a_p^2}{c^4},$$

Hyperbolic worldline in spacetime
is that of constant proper 4-acceleration!

Twin Paradox



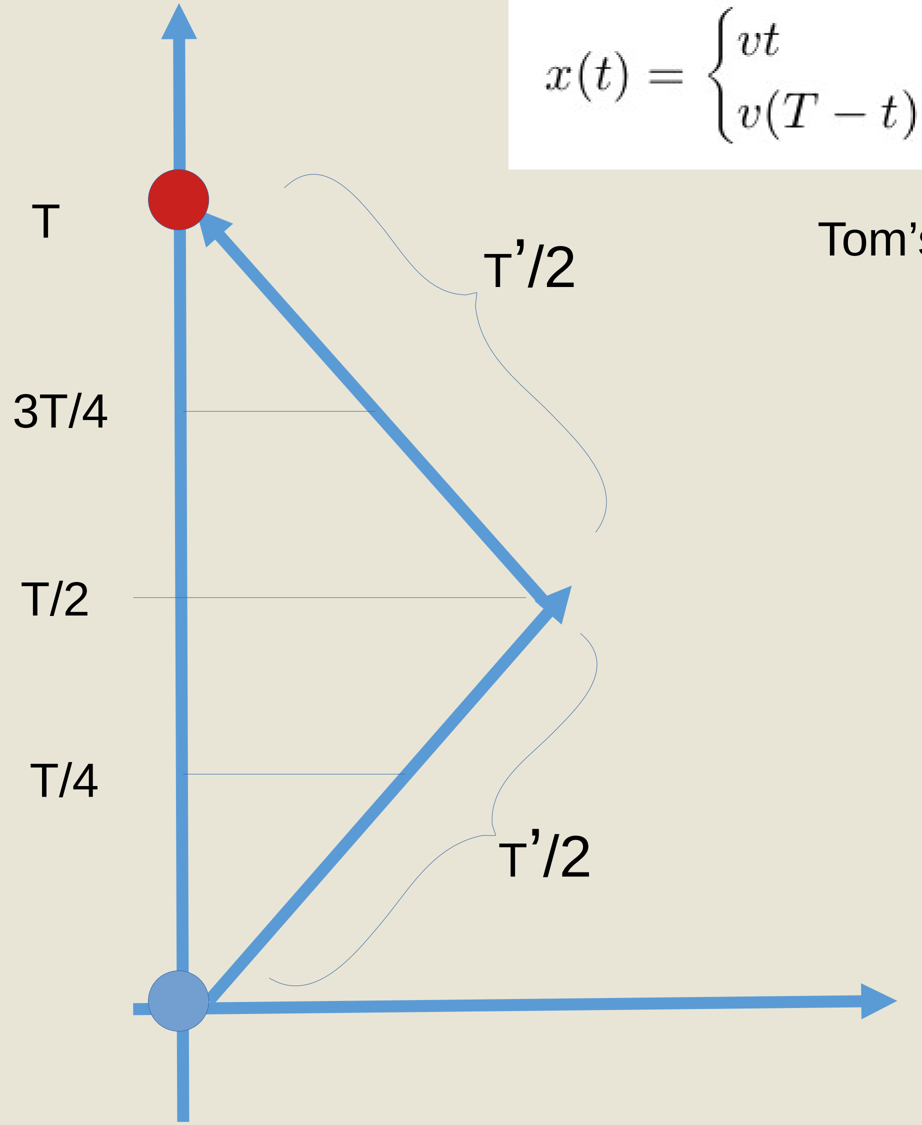
Tom and Bob are 21 years old

Tom travels at a speed 0.96c to a distant star. Instantaneously upon arrival, returns to earth at speed 0.96c.

Tom is 35 years old

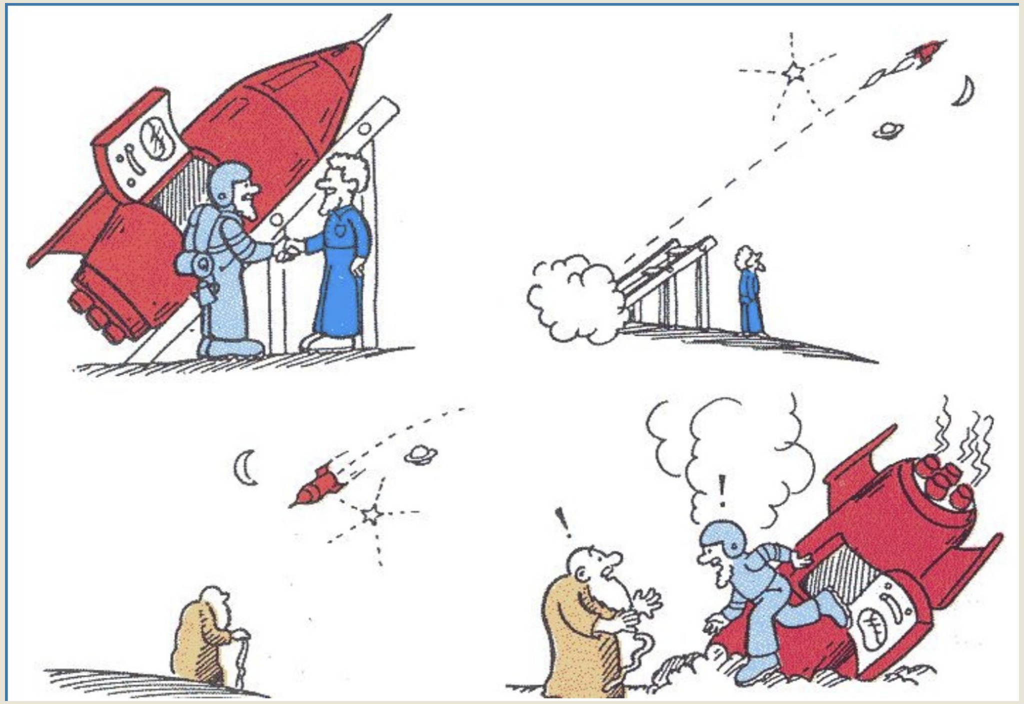
Bob is 71 years old!

Twin Paradox



$$x(t) = \begin{cases} vt & \text{if } t \in [0, T/2] \\ v(T-t) & \text{if } t \in [T/2, T] \end{cases}$$

Tom's worldline



$$dx = \pm v dt$$

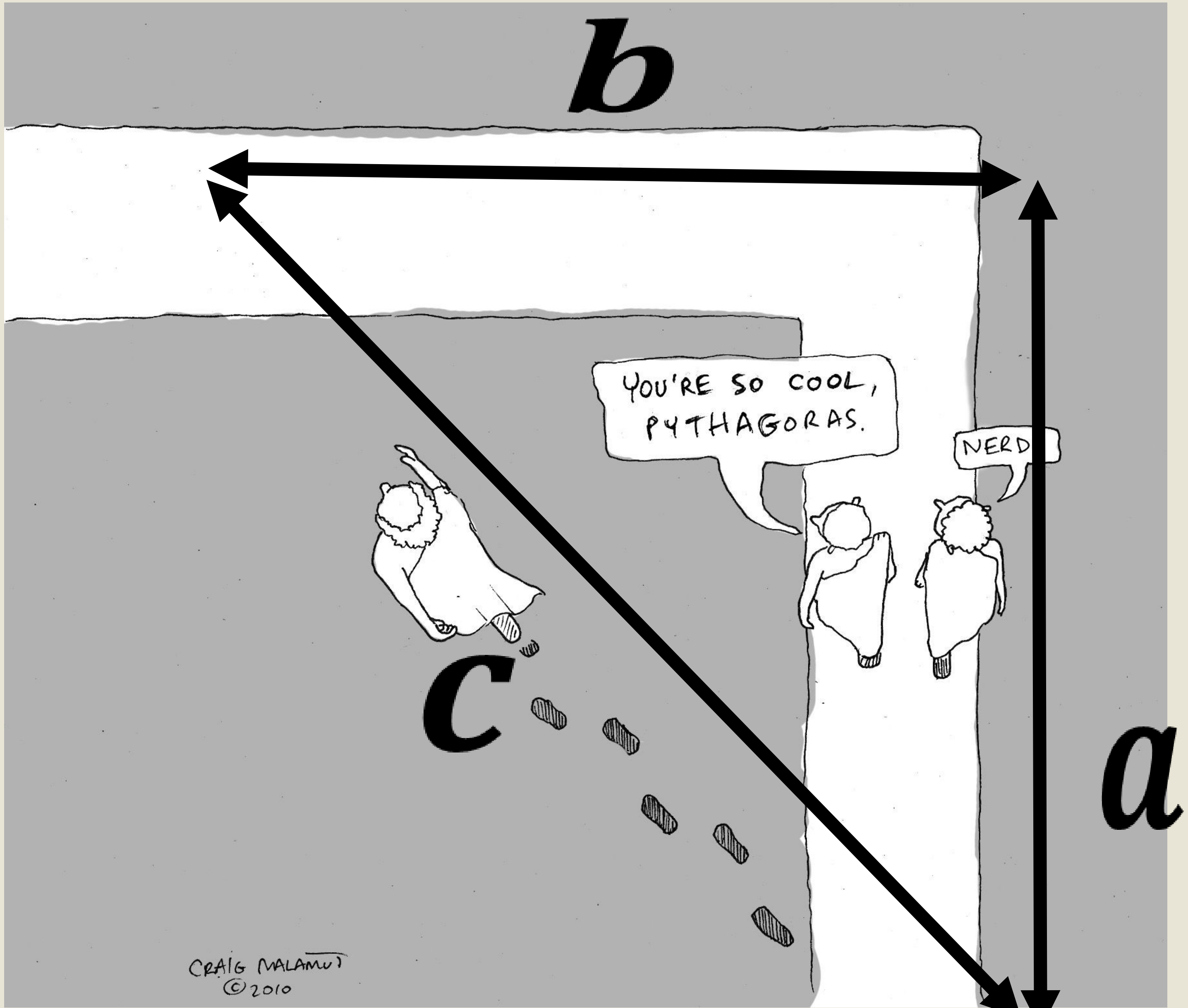
$$c d\tau = \sqrt{-||\dot{\boldsymbol{x}}||^2_{Mink}}$$

$$||\dot{\boldsymbol{x}}||^2_{Mink} = -c^2 dt + (\pm v dt)^2 \implies c\, d\tau = \sqrt{c^2 - v^2} dt$$

$$\frac{\Delta \tau}{\Delta t} = \sqrt{1 - \frac{v^2}{c^2}},$$

$$\frac{T'}{T} = \sqrt{1 - \frac{v^2}{c^2}} \leq 1$$

Triangle inequality (Euclidean space)

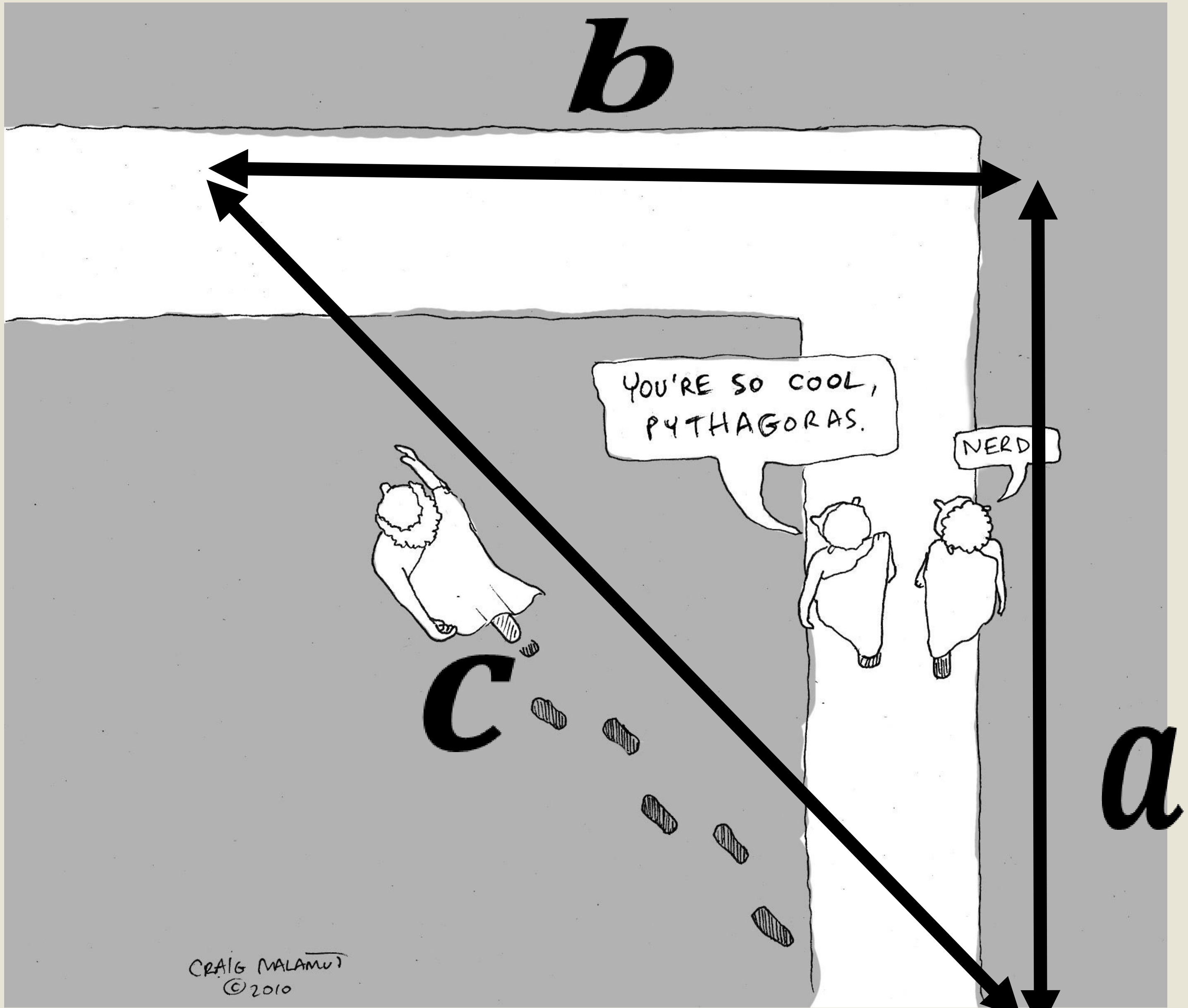


$$\mathbf{c} < \boldsymbol{a} + \boldsymbol{b}$$

Pithagoras takes the Shortest path

#Euclides

Reversed triangle inequality (Minkowski space)

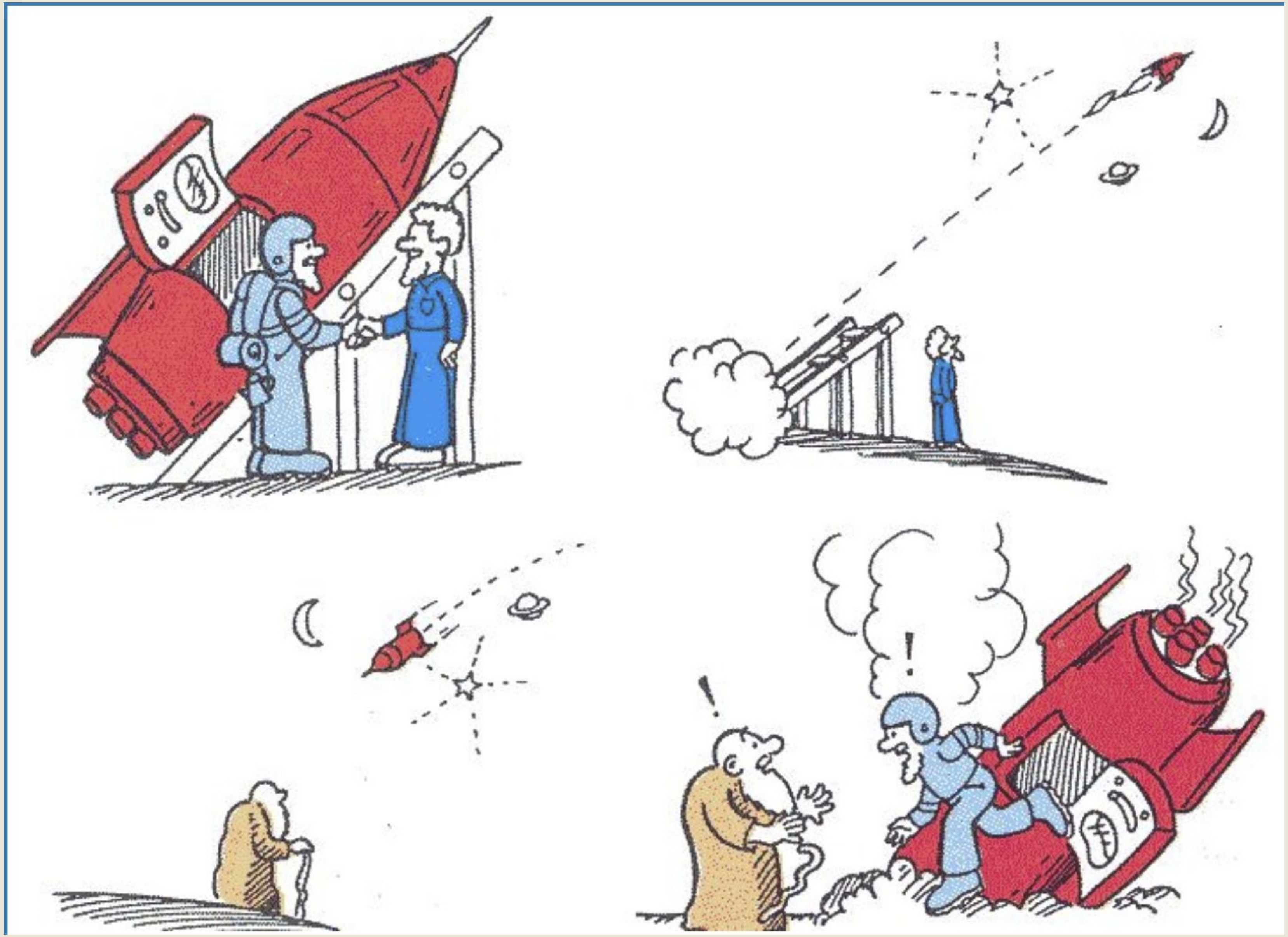


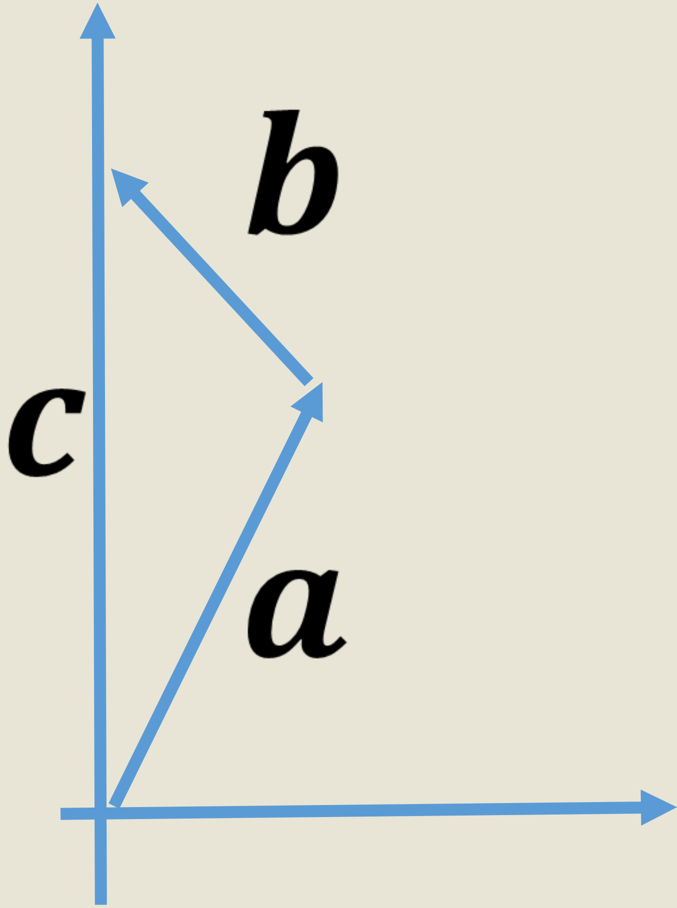
$$\mathbf{c} > \boldsymbol{a} + \boldsymbol{b}$$

Pithagoras takes the longest path

#Minkowski

Twin paradox as the reversed triangle inequality





$$c > a + b$$

Spike in curve = instant infinite acceleration

David Bowie as Major Tom

For each 6 months Major Tom spends in the ISS, major Tom is 0.007 seconds younger than Bob.

A realistic worldline: Langevin's travel

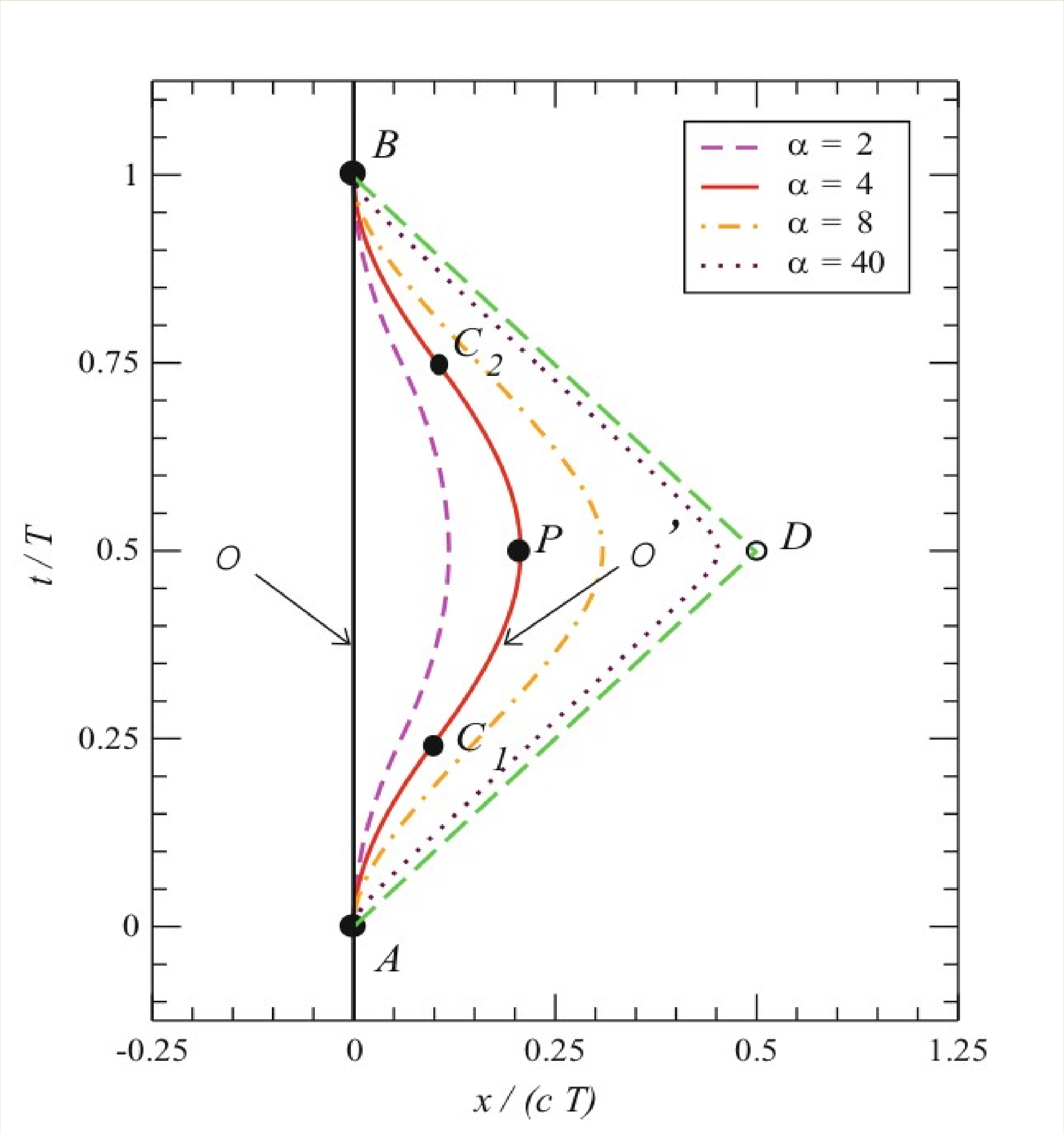


Image taken from Erik Gourgoulhon's book Special Relativity in General Frames

Wordline

$$\text{for } t \in \left[0, \frac{T}{4}\right]: \quad x(t) = \frac{cT}{\alpha}\left[\sqrt{1+\alpha^2 (t/T)^2} - 1\right] \tag{2.20a}$$

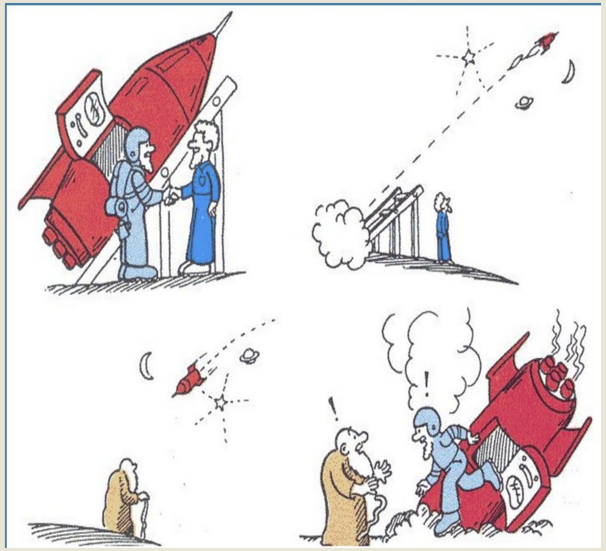
$$\text{for } t \in \left[\frac{T}{4}, \frac{3T}{4}\right]: \quad x(t) = \frac{cT}{\alpha}\left[-\sqrt{1+\alpha^2 (t/T - 1/2)^2} + 2\sqrt{1+\frac{\alpha^2}{16}} - 1\right] \tag{2.20b}$$

$$\text{for } t \in \left[\frac{3T}{4}, T\right]: \quad x(t) = \frac{cT}{\alpha}\left[\sqrt{1+\alpha^2 (t/T - 1)^2} - 1\right], \tag{2.20c}$$

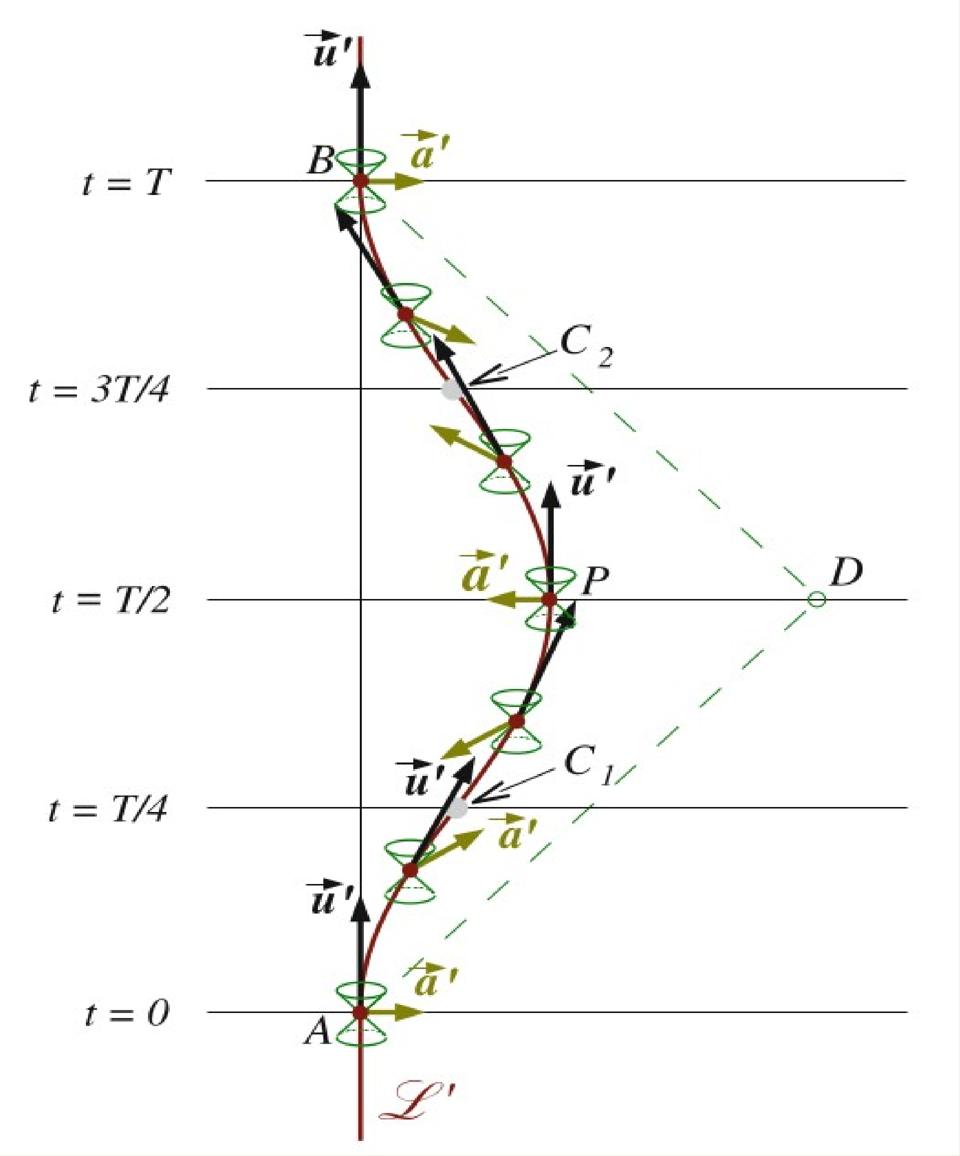
Eq. 2.20a rewritten

$$\left(\alpha \frac{x}{cT} + 1\right)^2 - \left(\alpha \frac{t}{T}\right)^2 = 1,$$

2.20a-b-c: three hyperbolas cut and joint



A realistic worldline: Langevin's travel



k=2

k=1

k=0

Compute worldline's tangent

$$\mathrm{d}x = (-1)^k \frac{\alpha(t/T - k/2)}{\sqrt{1+\alpha^2\,(t/T-k/2)^2}}\, c\, \mathrm{d}t,$$

Compute proper time $$c d\tau = \sqrt{-||\dot{\boldsymbol{x}}^2||_{Mink}}$$

$$c\, d\tau = \frac{dt}{\sqrt{1+\alpha^2(t/T-k/2)^2}}$$

Integrate this ODE to get $\tau = \tau(t)$

and evaluate at initial and final events

$$\frac{T'}{T} = \frac{\tau(B)-\tau(A)}{t(B)-t(A)} = \frac{4}{\alpha}\,\mathrm{arcsinh}\left[\frac{\alpha}{4}\right] \leq 1$$

Side comment: Physical acceleration in local reference frame $a_{\mathrm{RF}} = \alpha\,\frac{c}{T}$

Image taken from Erik Gourgoulhon's book
Special Relativity in General Frames

Mach’s principle and rotation: Absolute or Relative?

Taken from The Forgotten Mystery of Inertia. American Scientiest.

Rotation is absolute!

Interestellar’s docking scene.

Rotation is Relative!

Mach’s principle and Einstein’s road to General Relativity

Mach’s principle:
inertial forces experienced by a body in nonuniform motion are determined by the quantity and distribution of matter in the universe.

Connection
between geometry and matter?

Taken from The Forgotten Mystery of Inertia by Tony Rothman. American Scientiest.

General Relativity

“Matter tells space how to bend,
space tells matter how to move”

J.A. Wheeler

Geodesic equation

$$\nabla_{\dot{\boldsymbol{x}}} \dot{\boldsymbol{x}} = 0$$

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0$$

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