Tensors in physics

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Motivation

- Why care about tensors?
  - A standard language in general relativity, theoretical physics, mathematics
  - New applications in machine learning, quantum phys./inf. (tensor networks)
- What does it offer you?
  - Partial Differential Equations (PDEs) are ubiquitous in physics
  - Tensors provide formalism for PDEs in arbitrary coordinate system...
  - ...that also allows easy translation to explicit calculation

What's a tensor?

- Circular definition:

> A tensor is something that transforms like a tensor

- Kid definition:

> Tensors "eat" vectors, then "spit out" a scalar in a linear way

- Adult definition:

> Tensors are multilinear maps
>
> - Let $\mathbb{V}$ be vector space and $\mathbb{V}^*$ be its dual, defined over field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$)
> - Tensors are multilinear maps T
>
> $$T : \mathbb{V} \times \mathbb{V} \times \ldots \times \mathbb{V} \times \mathbb{V}^* \times \mathbb{V}^* \ldots \times \mathbb{V}^* \to \mathbb{F}$$

Linear maps

- Let $u, v \in \mathbb{V}$ be vectors, and $a, b \in \mathbb{R}$ be real numbers.
- Suppose w is map which "eats" vector and "spits out" real number: $w : \mathbb{V} \to \mathbb{R}$
- Linearity means:

$$w(a\,u + v) = a\,w(u) + w(v)$$

  If w satisfies above, it is a linear map.

- Linear maps are simple examples of tensors

Dual vectors* as linear maps

- Dual vectors $w \in \mathbb{V}^*$ are linear maps $w : \mathbb{V} \to \mathbb{R}$
- Given $v = (v^1, v^2, ..., v^N) \in \mathbb{V}$, can write linear map explicitly as

$$w(v) = \sum_{i=1}^{N} w_i \, v^i \qquad\qquad \text{2D Ex.:} \;\; w(v) = w_1 v^1 + w_2 v^2$$

- Equivalently, can describe linear map as coefficient list: $w = (w_1, w_2, ..., w_N)$.
- For finite dimension, linear maps w form a vector space $\mathbb{V}^*$ with same dim. as $\mathbb{V}$

*Alternative names: **one-forms**, **covariant vectors**

Vectors* as linear maps

- Vectors $v \in \mathbb{V}$ define linear maps $v : \mathbb{V}^* \to \mathbb{R}$

$$v(w) = \sum_{i=1}^{N} w_i \, v^i = w(v)$$

- **Exercise**: For dual vectors w, z, ... $\in \mathbb{V}^*$ convince yourself that $v(w)$ is linear:

$$v(a\,w + z) = a\,v(w) + v(z)$$

- Both vectors and dual vectors define linear maps

*Alternatively: **contravariant vectors**

Tensors

- Tensors are multilinear maps $T : \mathbb{V} \times \mathbb{V} \times \ldots \times \mathbb{V} \times \mathbb{V}^* \times \mathbb{V}^* \ldots \times \mathbb{V}^* \to \mathbb{R}$
- A tensor "eats" a bunch of vectors u, v, ... $\in \mathbb{V}$ and/or dual vectors w, z, ... $\in \mathbb{V}^*$, and "spits out" a real number in a the following way:

$$T(u, v, \ldots, w, z, \ldots) = \sum_{i=1}^{N} \sum_{j=1}^{N} \ldots \sum_{k=1}^{N} \sum_{l=1}^{N} \ldots \left[T_{ij\ldots}{}^{kl\ldots} \times u^i v^j \ldots \times w_k z_l \ldots \right]$$

- Simple examples:

$$L(u, v) = \sum_{i=1}^{N} \sum_{j=1}^{N} L_{ij}\, u^i\, v^j \qquad\qquad M(w, v) = \sum_{i=1}^{N} \sum_{j=1}^{N} M^i{}_j\, w_i\, v^j$$

- T, L, M are linear in each argument (**exercise**); they are *multilinear* maps

Einstein's "greatest" contribution

- Problem: Too many summation symbols
- Einstein's observation: summation indices in linear maps come in pairs
- Einstein summation convention:

  Any time you see a pair of indices (one raised and one lowered) written with the same symbol, a sum is implied.

$$w_i\, v^i = \sum_{i=1}^{N} w_i\, v^i$$

$$L_{ij}\, u^i\, v^j = \sum_{i=1}^{N} \sum_{j=1}^{N} L_{ij}\, u^i\, v^j \qquad M^j{}_i\, v^i = \sum_{i=1}^{N} M^j{}_i\, v^i$$

Coordinate transformations

- Consider coordinates $x = (x^1, x^2)$ and $y = (y^1, y^2)$ on 2D Euclidean space
- Coordinate transformation $x(y)$ can be written:

$$x(y) = \left(\; x^1(y^1, y^2) \;\; , \;\; x^2(y^1, y^2) \; \right)$$

- We typically assume transformation can be inverted: $y(x)$ exists*
- Can define Jacobian matrices:

$$\frac{\partial y^a}{\partial x^i} \Rightarrow \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^1}{\partial x^2} \\ \frac{\partial y^2}{\partial x^1} & \frac{\partial y^2}{\partial x^2} \end{pmatrix} \qquad \frac{\partial x^i}{\partial y^a} \Rightarrow \begin{pmatrix} \frac{\partial x^1}{\partial y^1} & \frac{\partial x^1}{\partial y^2} \\ \frac{\partial x^2}{\partial y^1} & \frac{\partial x^2}{\partial y^2} \end{pmatrix} \qquad \frac{\partial y^a}{\partial x^i}\frac{\partial x^i}{\partial y^b} \Rightarrow I$$

* Inverse function theorem: coordinate transformations are invertible near a point if determinant of Jacobian at point is nonsingular

Transformation of vectors

- Scalar function $\phi(x) = \phi(x^1, x^2)$
- Can rewrite as $\phi'(y) = \phi(x(y))$
- Directional derivative $\vec{v} \cdot \vec{\nabla}\phi$ is scalar:

$$\vec{v} \cdot \vec{\nabla}\phi = v^i \frac{\partial \phi}{\partial x^i}$$

- Chain rule:

$$\frac{\partial \phi}{\partial x^i} = \frac{\partial y^a}{\partial x^i} \frac{\partial \phi'}{\partial y^a}$$

- Directional derivative in new coordinates:

$$\vec{v}' \cdot \vec{\nabla}'\phi' = v'^a \frac{\partial \phi}{\partial y^a}$$

- Value of a scalar independent of coordinates (at a point)
- Demand $\vec{v} \cdot \vec{\nabla}\phi = \vec{v}' \cdot \vec{\nabla}'\phi'$

$$v^i \frac{\partial y^a}{\partial x^i} \frac{\partial \phi'}{\partial y^a} = v'^a \frac{\partial \phi'}{\partial y^a}$$

Transformation laws

- Vectors must transform as:

$$v'^a = \frac{\partial y^a}{\partial x^i} v^i$$

- Demand $w(v) = v^i w_i = v'^a w'_b$

$$w'_a = \frac{\partial x^i}{\partial y^a} w_i$$

- Follows from:

$$\frac{\partial x^i}{\partial y^a}\frac{\partial y^a}{\partial x^j} = \delta^i{}_j \qquad \frac{\partial y^a}{\partial x^i}\frac{\partial x^i}{\partial y^b} = \delta^a{}_b$$

- $\delta^i{}_j$ are components of identity matrix I

> $\delta^i{}_j = 1$ if $i = j$
> $\delta^i{}_j = 0$ if $i \neq j$

- For tensor $T(w, v)$, demand:

$$T(w,v) = T^i{}_j w_i v^j = T'^a{}_b w'_a v'^b$$

- For a tensor with components $T^i{}_j$:

$$\boxed{T'^a{}_b = \frac{\partial y^a}{\partial x^i}\frac{\partial x^j}{\partial y^b} T^i{}_j}$$

Tensor transformation laws: summary

- Tensor transformation law for components ${T^i}_j$ of T:

$$\boxed{{T'^a}_b = \frac{\partial y^a}{\partial x^i}\frac{\partial x^j}{\partial y^b}{T^i}_j}$$

- Raised indices of tensor components transform like vector indices
- Lowered indices of tensor components transform like dual vector indices
- Laws obtained from demand that linear maps (e.g. $T(w, v)$) transform like scalars
- Consequence: **Tensor equations have same form in all coordinate systems**

$${A^i}_j = {B^i}_j \qquad \Leftrightarrow \qquad {A'^a}_b = {B'^a}_b$$

Application: Basic geometry

- Metric tensor ($g : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{R}$), written as g_{ij}
- Generalizes dot product: $u \cdot v \quad \Rightarrow \quad g(u, v) = g_{ij} u^i v^i$
- Generalizes Pythagorean thm:

$$\Delta s^2 = \Delta x^2 + \Delta y^2 \quad \Rightarrow \quad ds^2 = g_{ij} dx^i dx^j$$

- Can measure distances along parameterized curves $x^i(t)$ by integrating:

$$\int ds = \int \sqrt{g_{ij} dx^i dx^j} = \int \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}} dt$$

- Metric can also be used to turn vector into dual: $v_i = g_{ij} v^j$

Application: Partial Differential Equations

- Partial derivatives (PDs) of scalars are tensors, but PDs of vectors and tensors are not tensors
- Can construct a covariant derivative operator ∇_i which yields a tensor

$$\nabla_k T^i{}_j = \frac{\partial T^i{}_j}{\partial x^k} + \Gamma^i_{kl} T^l{}_j - \Gamma^l_{kj} T^i{}_l$$

- The catch? Coefficients Γ^i_{jk} do not transform like tensors:

$$\Gamma^a_{bc} = \left(\frac{\partial y^a}{\partial x^i} \frac{\partial x^j}{\partial y^b} \frac{\partial x^k}{\partial y^c} \right) \Gamma^i_{jk} - \left(\frac{\partial x^j}{\partial y^b} \frac{\partial^2 y^a}{\partial x^j \partial x^i} \right) \frac{\partial x^i}{\partial y^c}$$

- Often, can translate PDEs to tensor form by replacing PDs with ∇_i

Application: tensor networks

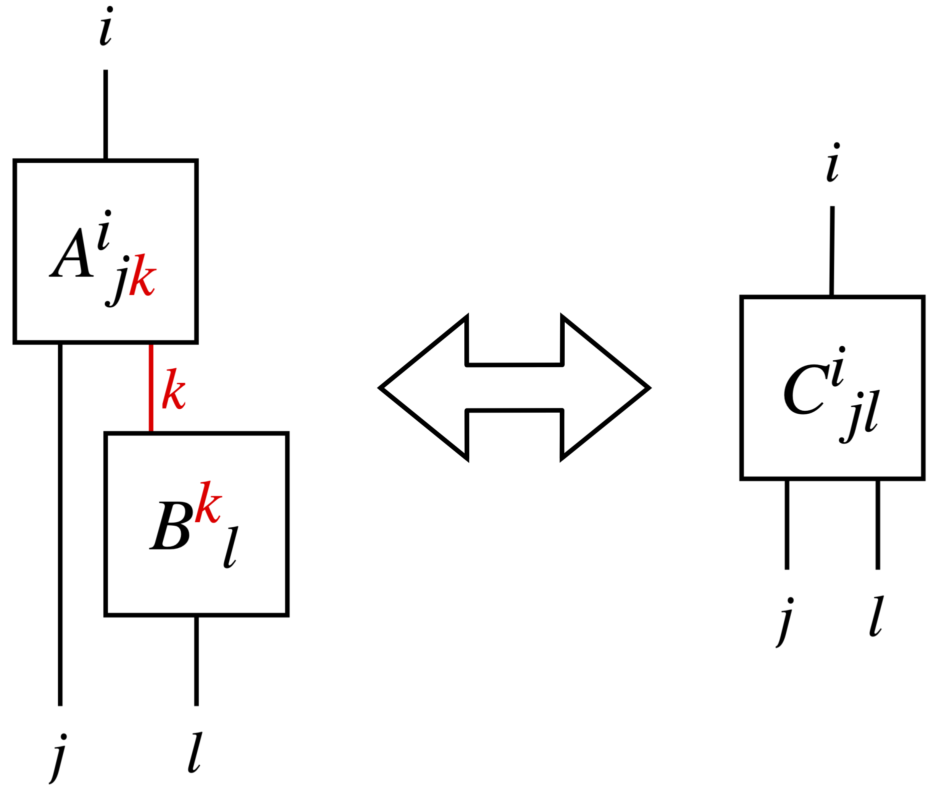
- Contraction means summing over indices (one raised one lowered):

$$C^i{}_{jl} \quad \Rightarrow \quad C^i{}_{ji} = C_j$$

- Tensors can also be combined to create new tensors:

$$A^i{}_{jk} B^k{}_l = C^i{}_{jl}$$

- Can describe diagrammatically (Penrose)
- Networks of tensor contractions can describe factorization of linear operators



Stuff to read

- Carroll, *Spacetime and Geometry* , Schutz, *A First course in GR*
- Wald, *General Relativity* (For learning tensors and GR properly)
- Synge and Schild, *Tensor Calculus* (For an old-fashioned approach)
- Penrose and Rinder, *Spinors and Spacetime* (For Penrose notation)
- Biamonte and Bergholm, *Tensor networks in a nutshell*, arXiv:1708.00006
- My notes on tensors (Read these in secret, pretend you learned from Wald):
  Poor man's introduction to tensors
  https://justincfeng.github.io/Tensors_Poor_Man_2c.pdf