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Gravitational collapse

based on J. R. Oppenheimer and H. Snyder, Physical Review **56**, 455 (1939)

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Introduction

- In 1916, Schwarzschild published the first non-trivial solution of the theory of General Relativity.

  ↳ The Schwarzschild solution describes the spacetime outside a point mass.

- Although it would take many decades to fully understand the Schwarzschild solution, already in 1932, after the work of Hilbert, Eddington and Lemaître, it was clear that this solution contained a non-removable singularity at the position of the point mass.

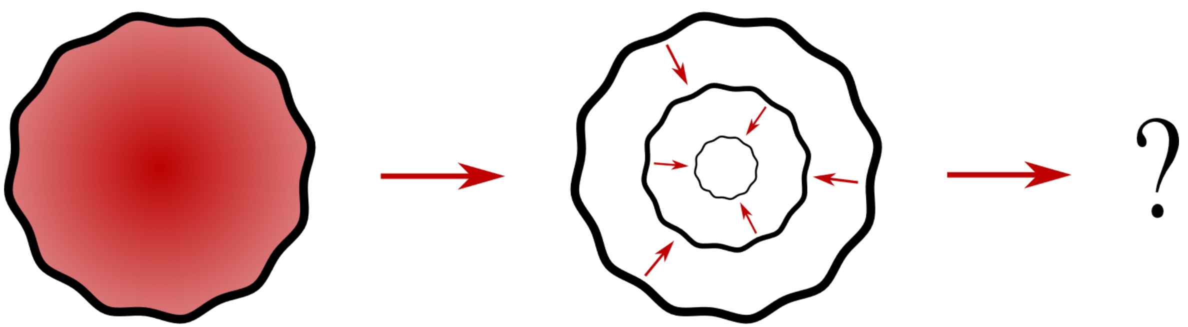
- This led to various debates about the solution and the theory itself. However, by 1939 the consensus was that the Schwarzschild solution would not result from the evolution of physical initial data.

Introduction

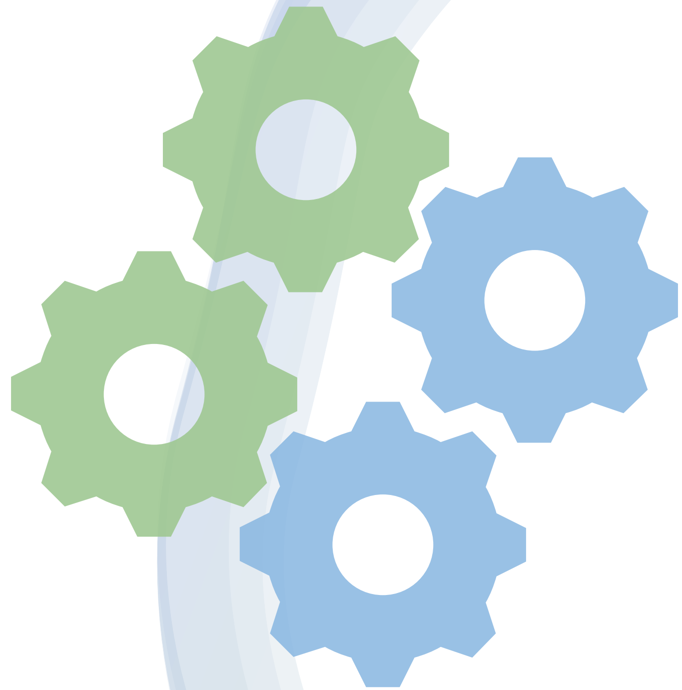
- In 1939, Oppenheimer and Volkoff showed that there is a maximum mass beyond which there are no static solutions of the theory of General Relativity.

  ↳ The original value is quite deprecated and currently it is predicted that cold neutron stars with more than 2.9 solar masses can not exist.

- What happens to massive compact objects with masses above this threshold when they run out of nuclear sources of energy and start to collapse under their weight?



Setup



Setup

- Finding a general global solution for a collapsing star with proper boundary conditions is hard. Then, to study the evolution of gravitational collapse, we are going to consider a simple model.

↳ We will consider two solutions of the EFE matched at a common boundary.

↳ We will then have a solution to model the interior of the star and a solution for the outside of the star.

↳ Provided that certain conditions are verified at the boundary, the total (combined) spacetime is a solution of the EFE.

↳ We will also consider that no matter or radiation is ejected by the star and impose that the system is spherical symmetric at all times.

The interior solution

The interior solution

- For the interior, consider a spherically symmetric solution of the EFE permeated by dust.

  - That is, the star is composed by particles that only interact gravitationally and the thermodynamic pressure is negligible.

  - Such fluid is characterized by a stress-energy tensor of the form

$$T^{\alpha\beta} = \rho u^{\alpha} u^{\beta} \,,$$

    where ρ represents the energy density of the fluid and u^{α} is the 4-velocity of an element of volume of the fluid.

- Moreover, we will impose that the star is homogeneous, that is ρ is independent of the spatial coordinates.

The interior solution

- Under these assumptions, the EFE can be solved and we find that the spacetime is characterized by the following line element

$$ds_i^2 = -d\tau^2 + a(\tau)^2 \left(d\chi^2 + \sin^2\chi d\Omega^2\right) \ ,$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ represents the line element of the unit 2-sphere, $\chi \in [0, \pi]$ and the function a verifies

$$\dot{a}^2 + 1 = \frac{b}{a} \,,$$

where $b \in \mathbb{R}_{>0}$.

- Moreover, we also find

$$\rho a^3 = \frac{3b}{8\pi} = \text{constant} \,.$$

The interior solution

- Imposing that at the initial time: $\tau = 0$, the function a has a local maximum, that is

$$a\left(0\right) = a_{max} \ ,$$

we can integrate the equation

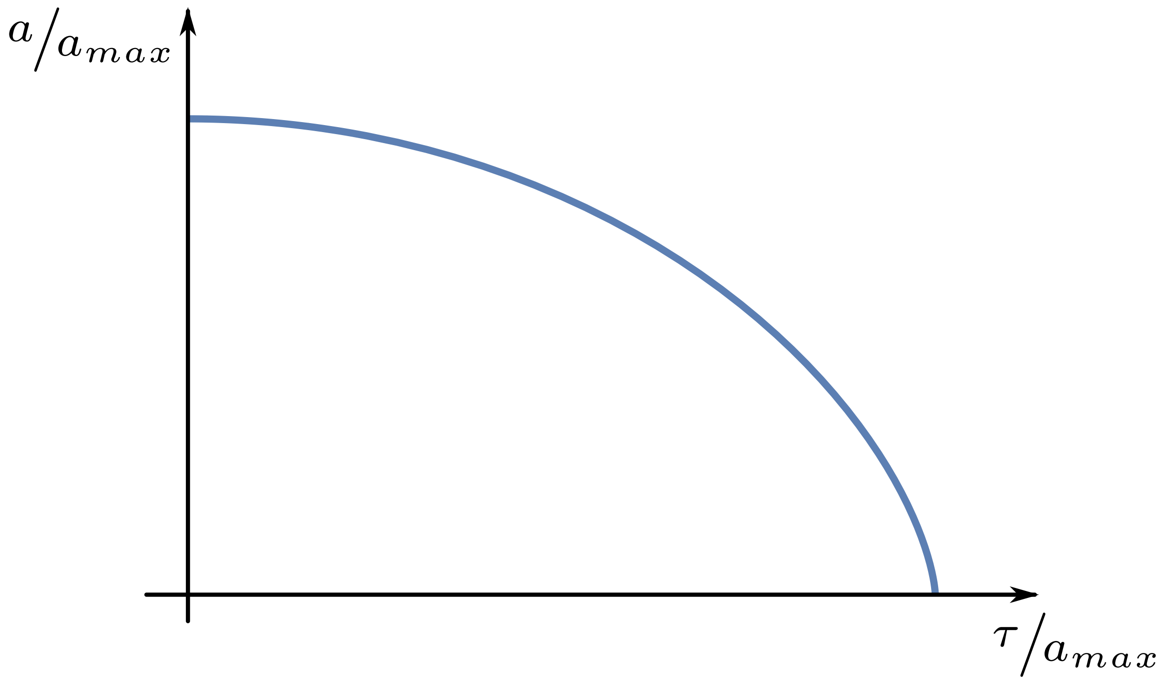
$$\dot{a}^2 + 1 = \frac{b}{a} \, ,$$

finding, in parametric form:

$$\begin{cases} \tau\left(\eta\right) = \dfrac{1}{2} a_{max}\left(\eta + \sin\eta\right) \\ a\left(\eta\right) = \dfrac{1}{2} a_{max}\left(1 + \cos\eta\right) \end{cases}$$

The interior solution

$$\begin{cases} \tau(\eta) = \frac{1}{2} a_{max} (\eta + \sin\eta) \\ a(\eta) = \frac{1}{2} a_{max} (1 + \cos\eta) \end{cases}$$



- The collapse starts at $\eta = 0$, where $a = a_{max}$, and ends at $\eta = \pi$ where $a = 0$.

The exterior solution

The exterior solution

- Outside the star, we will consider that the spacetime is vacuum and spherically symmetric.

  ↳ Then, the spacetime is described by the Schwarzschild solution.

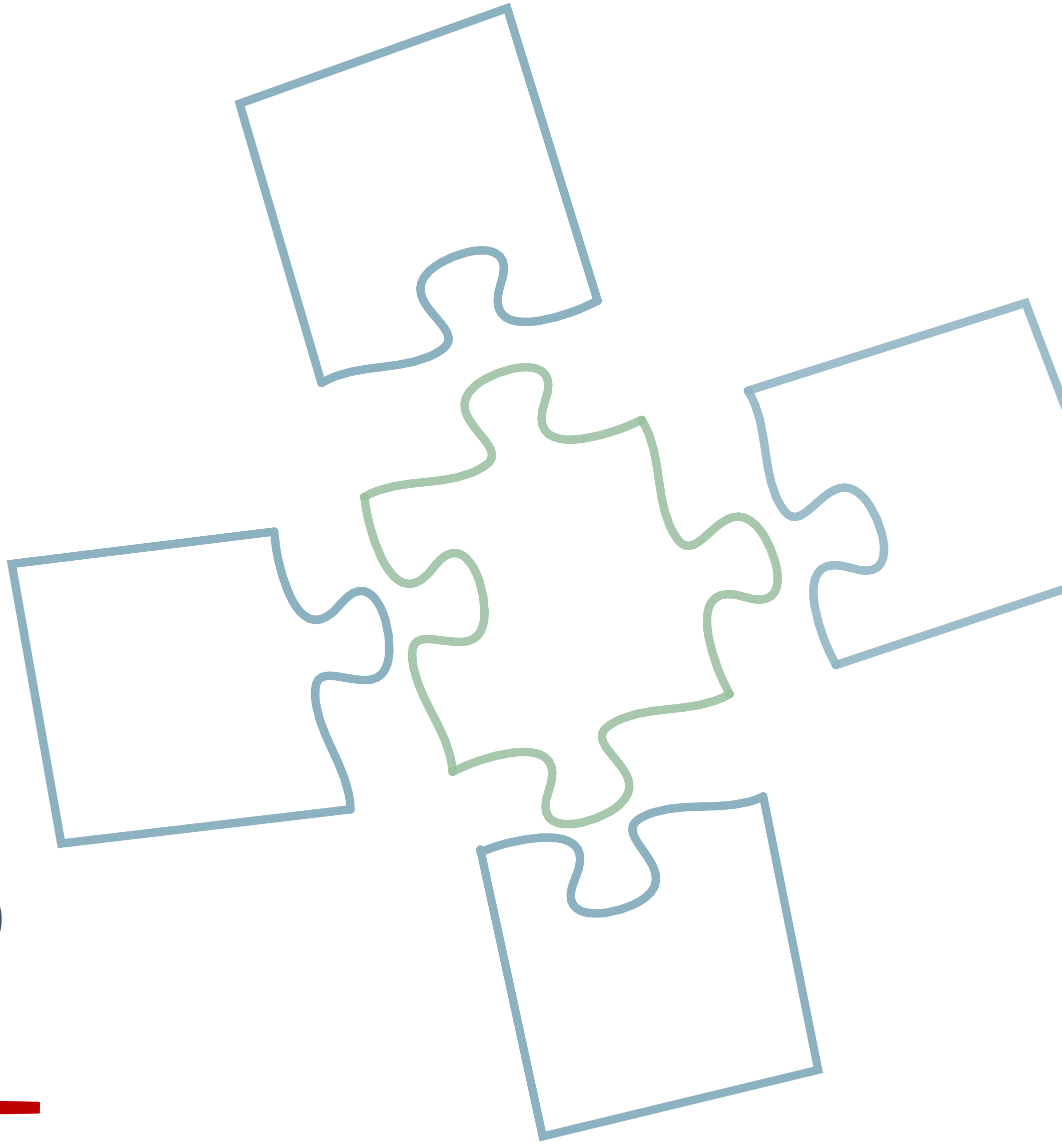
- The Schwarzschild solution is characterized, in spherical coordinates (r, θ, φ), by the line element

$$ds_e^2 = -f(r)\, dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega^2 \,,$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ represents the line element of the unit 2-sphere and

$$f(r) = 1 - \frac{2M}{r} \,.$$

Matching the two solutions



Matching the two solutions

- The two previous solutions describe spatially infinite spacetimes.

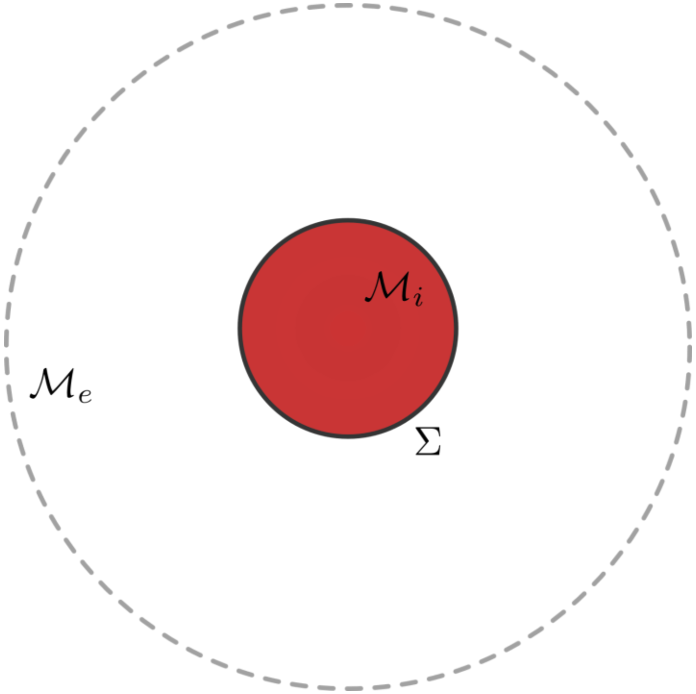
- To use those solutions to build a model for a collapsing massive compact object, we will truncate each solution, such that:

  - the solution for the interior will be considered only to a maximum value $\chi = \chi_0 < \pi$;

  - in the solution for the exterior, we will consider the region from infinity to a hypersurface with circumferential radius $r = R > 0$.

Matching the two solutions

- The truncated solutions will be smoothly matched at a common hypersurface, say Σ, such that the spacetime is composed by the interior and exterior solutions.



↳ However, what are the conditions for the total spacetime to be a solution of the theory of General Relativity?

Matching the two solutions

- This cut and paste procedure to find solutions from matching two known solutions is called the Israel-Darmois junction formalism.

- To understand the conditions imposed by the Israel-Darmois formalism, we need to introduce two quantities that characterize a spacetime embedded in a higher dimensional spacetime: the induced metric and the extrinsic curvature.

The induced metric

- Consider a 4-dimensional spacetime $\mathcal{M}$ with metric g. Also, consider a 3-dimensional hypersurface Σ that is a subspacetime of $\mathcal{M}$,
- Then, the geometry of the higher dimensional spacetime, $\mathcal{M}$, induces an intrinsic geometry on Σ, that is, the metric tensor g induces a metric tensor on Σ, say h.

↳ The metric h is called the induced metric of Σ.

↳ The tensor h completely characterizes the intrinsic geometry of the hypersurface Σ.

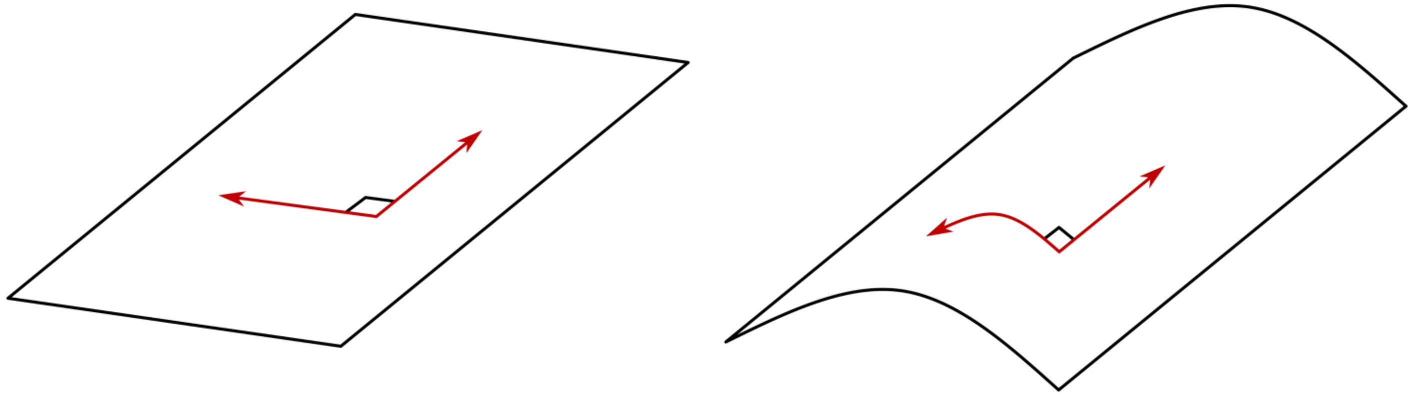
The extrinsic curvature

- On the other hand, to characterize the geometry of the hypersurface Σ, as perceived by the ambient 4-dimensional space $\mathcal{M}$ we need another quantity: the extrinsic curvature K.

- The extrinsic curvature K describes how Σ "curves" as seen from the higher dimensional spacetime $\mathcal{M}$.

The extrinsic curvature

$\mathbb{R}^3$



The matching conditions

- The Israel-Darmois junction formalism states that two solutions of the EFE can be smoothly matched at a common hypersurface Σ, so that the total spacetime is a valid solution of the field equations, if and only if

  - The induced metric of Σ, induced by each spacetime, is the same:

$$h^{-} = h^{+}$$

  - The extrinsic curvature of Σ as seen from each spacetime is the same:

$$K^{-} = K^{+}$$

- In our model, the hypersurface Σ represents the surface of the star.

Matching the two solutions

- To match the two solutions, it is convenient to chose the coordinates on Σ, the matching hypersurface, to be (τ, θ, φ).

- The induced metric on Σ, as seen from the interior spacetime, is

$$ds^2_{\Sigma^-} = -d\tau^2 + a\left(\tau\right)^2 \sin^2 \chi_0 d\Omega^2$$

- The induced metric on Σ, as seen from the exterior spacetime, is

$$ds^2_{\Sigma^+} = -\left[f\left(R\right)\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{f\left(R\right)}\left(\frac{dR}{d\tau}\right)^2\right]d\tau^2 + R^2\left(\tau\right)d\Omega^2 ,$$

where

$$f\left(R\right) = 1 - \frac{2M}{R} .$$

The global solution

- Then, imposing the junction conditions yields

$$R(\tau) = a(\tau)\sin\chi_0 \,,$$

$$f(R)\frac{dt}{d\tau} = \cos\chi_0 \,,$$

$$f(R)\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{f(R)}\left(\frac{dR}{d\tau}\right)^2 = 1 \,.$$

- The first equation determines the behavior of the circumferential radius of the surface of the star as a function of the proper time, τ, of an observer comoving with Σ.

  ↳ So, an observer comoving with the surface of the star will see the star collapse to a point in finite (proper) time.

The global solution

- An observer at spatial infinity describes the evolution of the surface of the star by a function $R(t)$.

- Using the remaining equations

$$f(R)\,\frac{dt}{d\tau} = \cos\chi_0\,,$$

$$f(R)\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{f(R)}\left(\frac{dR}{d\tau}\right)^2 = 1\,,$$

we find

$$\frac{dt}{dR} = -\frac{1}{f(R)\sqrt{1-\frac{f(R)}{\cos^2\chi_0}}}$$

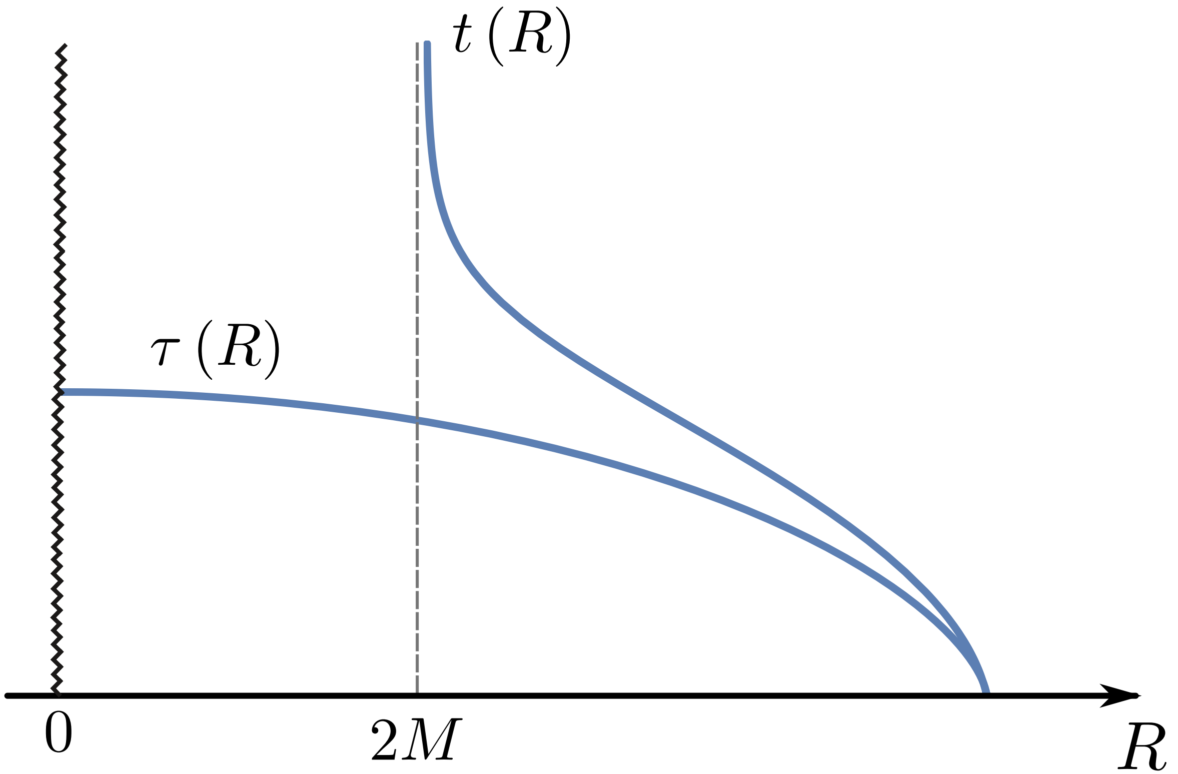
The global solution

- Integrating the last equation yields

$$\frac{1}{2M}t\left(R\right) = \mathrm{cotan}\chi_0\sqrt{\frac{R}{2M}\left(1-\frac{R}{2M}\sin^2\chi_0\right)} + 2\mathrm{arctanh}\left(\sqrt{\frac{R\cos^2\chi_0}{2M-R\sin^2\chi_0}}\right)$$
$$-\frac{\left(2\cos^2\chi_0-3\right)\cos\chi_0}{\left(\cos^2\chi_0-1\right)^{\frac{3}{2}}}\mathrm{arcsinh}\left(\sqrt{\frac{\cos^2\chi_0-1}{2M}R}\right)+C\,,$$

where C is an integration constant.

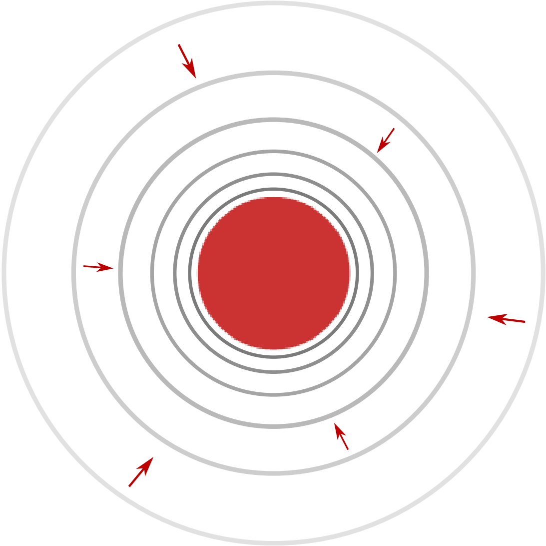
The global solution



Frozen stars or black-holes?

- As seen by an outside observer, the collapsing star seems to freeze in time.

  ↳ This led to the name of *Frozen Stars*.



- However, if we point a telescope to a collapsing massive compact object, we will not see it gradually stop and freeze in time.

Frozen stars or black-holes?

- Consider an observer comoving with the surface of the collapsing star emitting a (monochromatic) light to spatial infinity.

- Assuming the emitted electromagnetic wave has frequency ν_s in the frame of the comoving observer, from the Schwarzschild line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2 d\Omega^2 ,$$

we find the following expression for the gravitational red-shift

$$\nu_\infty = \sqrt{1 - \frac{2M}{R}} \nu_s ,$$

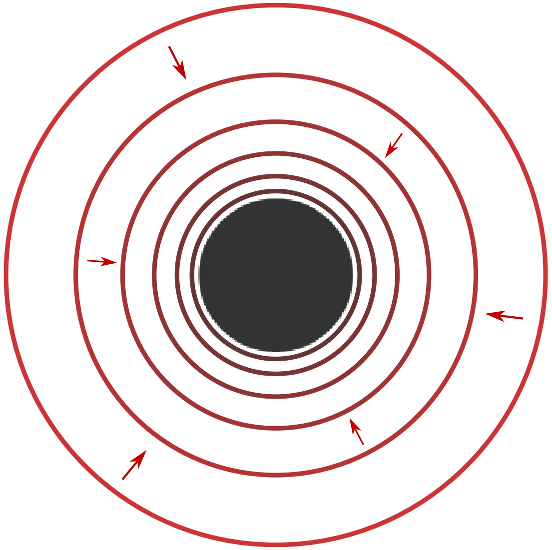
where ν_∞ represents the frequency of the wave in the frame of an observer at rest at spatial infinity

Frozen stars or black-holes?

$$\nu_\infty = \sqrt{1 - \frac{2M}{R}}\nu_s$$

- From this relation, we conclude that an observer at spatial infinity will, at some point, stop seeing the collapsing star.

  ↳ For an outside observer, the result of the gravitational collapse is a black object – a black-hole.



Summary

- Constructed a simple model for gravitational collapse by matching two known solutions of the theory of General Relativity.
- The interior solution described a homogeneous and isotropic dust cloud.
- The exterior spacetime was described by the Schwarzschild solution.
- Introduced the junction conditions for a smooth matching between the solutions.
- Analyzed the collapse from the point of view of an observer comoving with the surface of the star and an observer at rest at spatial infinity.
  - From the point of view of the comoving observer, the star will continuously collapse to a point.
  - From the point of view of the observer at infinity, the star will continuously collapse, yet it will progressively stop as it approaches the surface with $r = 2M$.