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Approximation Methods in the Study of Gravitational-Wave Generation: From the Quadrupole to the ZFL

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Resumo

Nesta tese são estudados métodos aproximados para o cálculo da radiação gravitacional emitida em vários processos. A não linearidade das equações the Einstein dificulta a tarefa de encontrar soluções radiativas exactas, o que exige que se aborde o problema ou numericamente, ou com métodos aproximados. Nesta tese estudam-se métodos aproximados. Começa-se por rever a relatividade geral linearizada, sendo depois utilizados dois métodos diferentes para calcular a energia radiada, através de ondas gravitacionais, em diferentes processos. Na primeira parte da tese considera-se uma expansão para velocidades baixas, a aproximação quadrupolo-octopolo. Esta aproximação é utilizada para calcular a energia e momento radiados, em dimensões pares, em dois processos: uma partícula pontual a cair radialmente num buraco negro de Schwarzschild-Tangherlini, e duas partículas pontuais em órbita circular. Na última parte da tese considera-se uma aproximação diferente, o limite de frequência zero (ZFL). Este método dá uma aproximação para o espectro da radiação emitida a baixas frequências, e para velocidades arbitráriamente altas. Utiliza-se este método para estimar a energia radiada na colisão de duas partículas pontuais, sendo calculado também o momento radiado no caso de uma colisão frontal. Finalmente considera-se a aplicação deste método para descrever a colisão de dois buracos negros, discutindo-se a aplicabilidade da mesma.
Parte dos resultados obtidos durante esta tese figuram nas referências [1] e [2].

Palavras-chave: Relatividade geral; Radiação gravitacional; Dimensões extra; Limite de frequência zero; Colisão de buracos negros.

Abstract

This thesis deals with approximation methods in the study of gravitational radiation emission. The non-linearity of Einstein equations makes it difficult to find exact radiative solutions, so one must employ either numerical, or approximation methods. In this thesis we study the latter. The linearized theory of general relativity is reviewed, and two approximation methods are employed to compute the energy emitted, through gravitational waves, in different processes. Both techniques rely on a linearized scheme. In the first part of the thesis we consider a small velocity expansion, the quadrupole-octopole approximation. This method is used to compute the radiated energy and momentum, in higher (even) dimensional spacetimes, for two different systems: a point particle falling radially into a Schwarzschild-Tangherlini black hole, and for two particles in circular orbit. In the last part of the thesis a different approach is pursued, the Zero Frequency Limit (ZFL), which provides an approximation of the low-frequency spectrum, valid for arbitrarily high velocities. This method is then employed to estimate the radiated energy (and momentum for the case of a head-on collision) in a point particle collision, generalizing the known results for the case of a non head-on collision. Finally the applicability of the ZFL approach to describe the high energy collision of two black holes is discussed.

Part of the results obtained during this thesis appear in Refs. [1] and [2].

Keywords: General Relativity; Gravitational radiation; Extra dimensions; Zero frequency limit; High energy black hole collisions.

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Chapter 1

Introduction

This thesis is devoted to the study of approximation methods for understanding gravitational radiation. Gravitational waves are a prediction of the theory of General Relativity (GR), and one expects to finally detect them in the forthcoming years. GR will pass a crucial test if gravitational waves are detected, and if the observed waveforms match the predicted templates. The fact that gravitational waves interact very weakly with matter makes them both hard to detect, and a good tool for gravitational wave astronomy, as they remain practically unaltered in their journey from source to detector. Since there are now detectors operating at, or near design sensitivity, there is a pressing need for accurate templates for the waveforms emitted in the various physical processes. The lack of exact radiative solutions makes it necessary to pursue either numerical solutions, or approximations, to find waveforms for different processes.

When two bodies collide or scatter gravitational radiation is emitted, due to the changes in momentum involved in the process. The computation of the radiated energy is most of the times only possible numerically, however there are several approximations which allow one to estimate the emitted energy. Here we describe two such approximations, the quadrupole-octopole approximation, and the Zero Frequency Limit (ZFL). The first is a small velocity expansion of the metric perturbation induced by the particles, while the latter is a long wave-length approximation, valid for arbitrary velocities, which provides a good approximation of the emitted energy spectrum at low frequencies. The first method is derived in Chapter 2, using the extension to higher dimensions of a formula for the metric perturbation first derived by Press [3]. This approximation is applied to compute the energy and momentum radiated (at quadrupole-octopole order) by a point particle falling radially into a higher dimensional Schwarzschild-Tangherlini black hole, and by two particles in circular orbit. The second method is applied in Chapter 3 to study the collision of two point particles. We start by reviewing the head-on collision, and gravitational scatter of point particles, studied by Smarr in Ref. [4] where the radiated energy and momentum are computed. Then the ZFL calculation is generalized for collisions with a finite impact parameter. Finally this collision is used as a toy model to describe the high energy collision of two black holes, and the ZFL results are compared against numerical and perturbative calculations.

The thesis is divided in three Chapters, Chapter 1 reviews the linearized theory of gravity, Chapter 2 studies the quadrupole-octopole approximation in higher dimensions, and Chapter 3 studies the Zero Frequency Limit approximation. Part of Chapter 3 has been submitted for publication [1].

1.1 Conventions

Unless otherwise stated we use geometrical units, that is $G = c = 1$. Einstein's summation convention is assumed, that i, $a^\mu b_\mu \equiv \sum_\mu a^\mu b_\mu$. $O(A)$ stands for terms of order A.

Metric, Riemann tensor and Einstein equations

The signature of the metric $g = g_{\mu\nu}dx^\mu \otimes dx^\nu$ is $(-+...+)$ and, unless otherwise stated, we take the number of spacetime dimensions, D, to be 4. Latin indices vary from 1 to $D-1$, and the Greek ones vary from 0 to $D-1$, where 0 denotes the time component, and $1, ...D-1$ the spatial components. The Minkwoski metric in D–dimensions is denoted by $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, ..., +1)$.
The inner product of two vectors V, W is denoted by $V \cdot W \equiv g(V, W) = g_{\mu\nu}V^\mu W^\nu$. 3−vectors are distinguished by bold-face, and the inner product between two 3−vectors $\mathbf{V}, \mathbf{W}$ is denoted by $\mathbf{V} \cdot \mathbf{W} \equiv g_{ij}V^i W^j$.

Our conventions for the Riemann tensor $R^\mu{}_{\nu\rho\sigma}$ are such that, in local coordinates $\{x^\mu\}$,

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu{}_{\nu\sigma} - \partial_\sigma \Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\alpha\rho}\Gamma^\alpha{}_{\nu\sigma} - \Gamma^\mu{}_{\alpha\sigma}\Gamma^\alpha{}_{\nu\rho}\,, \tag{1.1}$$

where $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$, and $\Gamma^\mu{}_{\nu\rho}$ are the Christoffel symbols for the Levi-Civita connection, which are determined uniquely from the metric by

$$\Gamma^\mu{}_{\nu\rho} = \frac{1}{2} g^{\mu\lambda} \left(\partial_\nu g_{\rho\lambda} + \partial_\rho g_{\lambda\nu} - \partial_\lambda g_{\nu\rho} \right)\,. \tag{1.2}$$

Thus the Einstein equations are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi}{c^4} T_{\mu\nu}\,, \tag{1.3}$$

where $R_{\mu\nu} \equiv R^\alpha{}_{\mu\alpha\nu}$ is the Ricci tensor and $R \equiv g^{\mu\nu}R_{\mu\nu}$ is the scalar curvature (or the Ricci scalar).

Fourier transform

We denote the Fourier transform by a $\tilde{F}$, and use the following conventions:

$$\tilde{F}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt\, e^{i\omega t} F(t)\,, \tag{1.4}$$

$$F(t) = \int_{-\infty}^{+\infty} d\omega\, e^{-i\omega t} \tilde{F}(\omega)\,, \tag{1.5}$$

$$\tilde{F}(\mathbf{k}) = \int_{-\infty}^{+\infty} d^{D-1}\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}} F(\mathbf{x})\,, \tag{1.6}$$

$$F(\mathbf{x}) = \frac{1}{(2\pi)^{D-1}} \int_{-\infty}^{+\infty} d^{D-1}\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{F}(\mathbf{k}) \tag{1.7}$$

1.2 Linearized Gravity

In this Chapter we review known results[5, 6, 7, 8, 9] of linearized theory of gravity and gravitational wave generation. We focus on 4 dimensional spacetimes only, although in Chapter 2 we use the linearized theory of gravity in (even) D−dimensional spacetimes. The study of gravitational radiation in higher dimensional spacetimes can be found in [10] and is also discussed in Sec. 2.1.

In order to introduce the linearized approximation, let us go briefly through some aspects of General Relativity. The gravitational action is $S = S_{EH}[g_{\mu\nu}] + S_M[g_{\mu\nu}, \Psi]$, where $g_{\mu\nu}$ denotes the metric, S_{EH} is the Einstein-Hilbert action, S_M is the matter action and Ψ denotes the matter fields. The Einstein-Hilbert action is given by

$$S_{EH} = \frac{1}{16\pi} \int d^4x\, \sqrt{-\det[g]} R\,, \tag{1.8}$$

where R is the scalar curvature (see Sec. 1.1), $d^4x = dx\,dy\,dz\,dt$, and $\det[g]$ stands for the metric determinant, which is negative. The Einstein field equations are obtained by taking the variation of the action with respect to the dynamical variable $g_{\mu\nu}$, $\frac{\delta S}{\delta g^{\mu\nu}} = 0$. This yields the following field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}\,, \tag{1.9}$$

where $R_{\mu\nu}$ is the Ricci tensor (see Sec. 1.1), and the energy-momentum tensor $T_{\mu\nu}$ is defined by

$$\delta S_M = \frac{1}{2} \int d^4x \sqrt{-\det[g]}\, T^{\mu\nu} \delta g_{\mu\nu} \tag{1.10}$$

The Einstein tensor is defined as $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$.
Since for any symmetric connection[1], as the Levi-Civita connection, the Riemann tensor satisfies the Bianchi identities, i.e. $R^{\rho}{}_{\alpha\beta\gamma} + R^{\rho}{}_{\beta\gamma\alpha} + R^{\rho}{}_{\gamma\alpha\beta} = 0$, the Einstein equations impose the conservation of the energy-momentum tensor. This means that $\nabla_\mu T^{\mu\nu} = 0$, where ∇_μ stands for the covariant derivative.

General Relativity is invariant under diffeomorphisms[2]. This means that Einstein equations do not determine completely the metric, that is, if a certain $g_{\mu\nu}$ is a solution of Einstein equations then a metric $g'_{\mu\nu}$ related to the first by a diffeomorphism is also a solution. This is called the local gauge invariance of General Relativity, and is analogous to the gauge invariance of Maxwell equations. The ambiguity is removed by fixing a gauge, i.e., by choosing a particular coordinate system, which removes the spurious degrees of freedom. Note that invariance under diffeomorphisms also implies the conservation of the energy-momentum tensor. This symmetry is of particular importance to gravitational waves, as one must check that these waves cannot be gauged away by an appropriate coordinate transformation. In fact we can count the number of unphysical degrees of freedom in the following way. The metric being a symmetric 2-tensor has 10 independent components, and the 10 Einstein equations should suffice to determine completely the metric. However, not all of Einstein equations are independent, in fact, since there are four Bianchi identities, $\nabla_\mu G^{\mu\nu} = 0$, there are only 6 independent equations, which leaves us with 4 unphysical degrees of freedom in $g_{\mu\nu}$. This freedom is represented by the local representation of the diffeomorphism, $x'^{\mu}(x^{\mu})$.

The full Einstein equations consist on a set of 10 second-order, nonlinear, coupled partial differential equations for the metric, therefore finding exact radiative solutions is extremely complicated (although some radiative solutions are known, such as the C metric [11]). In this Section we review the linearized theory of gravity, which studies the weak-field radiative solutions of Einstein equations. This approach describes waves with energy and momentum small enough not to affect their own propagation, overcoming the difficulty created by the fact that the energy-momentum tensor of gravitational waves contributes to their own propagation. Since it is expected that the gravitational waves detected are of low intensity this approach is justified in practice, for the propagation of gravitational waves.
When the gravitational fields are weak, we can express the metric as the flat Minkowski metric, $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$, plus a small perturbation $h_{\mu\nu}$, such that $|h^{\mu\nu}| \ll 1$, and that $g_{\mu\nu}$ approaches $\eta_{\mu\nu}$ asymptotically,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,. \tag{1.11}$$

Keeping only the leading term in $h_{\mu\nu}$, the inverse metric is

$$g^{\mu\nu} \equiv \left[g_{\mu\nu}\right]^{-1} = \eta^{\mu\nu} - h^{\mu\nu}\,. \tag{1.12}$$

[1] A connection is said to be symmetric whenever the torsion, which in local coordinates is given by $\Gamma^{\mu}{}_{\nu\rho} - \Gamma^{\mu}{}_{\rho\nu}$, vanishes.

[2] I.e. invariant under differentiable bijective transformations with a differentiable inverse. This invariance means that if the Universe is represented by a pseudo-Riemannian manifold (M, g, Ψ) with matter fields Ψ, then (M, g, Ψ) and $(M, \phi^* g, \phi^* \Psi)$ represent the same physical situation, where the map ϕ is a diffeomorphism $\phi : M \to M$ and ϕ^* denotes the pullback by ϕ. This means that there are no preferred coordinate systems in GR, which is often stated as the generally covariance of GR.

In a coordinate frame where Eq. (1.11) holds, one can expand the Einstein field equations in powers of $h_{\mu\nu}$, keeping only the lowest term in $h_{\mu\nu}$, to find the equations of motion obeyed by the perturbation. Note that we can raise and lower indexes using $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$ instead of $g^{\mu\nu}$ and $g_{\mu\nu}$, since the corrections would be of higher order in the perturbation. Expanding the Christoffel symbols (1.2) to first order in $h_{\mu\nu}$ we get

$$\Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2}\eta^{\mu\lambda}\left(\partial_\nu h_{\rho\lambda} + \partial_\rho h_{\lambda\nu} - \partial_\lambda h_{\nu\rho}\right) . \tag{1.13}$$

Similarly the Ricci tensor is

$$R_{\mu\nu} = \frac{1}{2}\left(\partial_\sigma\partial_\nu h^\sigma_\mu + \partial_\sigma\partial_\mu h^\sigma_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}\right) , \tag{1.14}$$

where the d'Alembertian is simply given by $\Box \equiv \eta^{\rho\nu}\partial_\mu\partial_\nu$, and $h \equiv \eta^{\mu\nu}h_{\mu\nu}$ is the trace of the perturbation. The Ricci scalar is then $R = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h$, and the Einstein equations in first order are

$$G_{\mu\nu} = \frac{1}{2}\left(\partial_\sigma\partial_\nu h^\sigma_\mu + \partial_\sigma\partial_\mu h^\sigma_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h^{\mu\nu} + \Box h\right) = 4\pi T_{\mu\nu} , \tag{1.15}$$

where $T_{\mu\nu}$ is the energy-momentum tensor, calculated to zeroth order in the perturbation. As the energy-momentum must be small, in order to the weak field approximation to apply, higher order contributions to the energy-momentum tensor will not be considered. This means that the lowest order in the energy-momentum tensor is of the same order of magnitude as the perturbation. As a consequence the covariant conservation of the energy-momentum tensor $\nabla_\mu T^{\mu\nu} = 0$ becomes

$$\partial_\mu T^{\mu\nu} = 0 , \tag{1.16}$$

where we have taken the covariant derivative in the zeroth order.

Note that the linearized theory of gravity can also be applied to perturbations about some other background – not necessarily Minkowski space – by expanding the metric in the same way $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$. This means that if one wanted to expand about flat background but with a noneuclidean coordinate basis, one had to expand about $\left(g^{(0)}_{\text{flat}}\right)_{\mu\nu}$. All derivatives would then have to be replaced by covariant derivatives corresponding to the affine connection compatible with the metric $\left(g^{(0)}_{\text{flat}}\right)_{\mu\nu}$.

Gauge transformations

As pointed out in the previous Section General Relativity has diffeomorphism invariance, which is broken due to choice of a coordinate system in which (1.11) holds. However a residual gauge symmetry remains, which is analysed in this Section.

The choice of a frame where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ does not completely specify the coordinate system of the background, as there may be other coordinate systems in which the metric can be expressed as a flat background plus a small perturbation, with a different perturbation. We now study such coordinates systems, to find which is the remaining gauge invariance. Let (M_p, g) be the physical pseudo-Riemannian manifold, where the metric g obeys the Einstein equations. Consider a diffeomorphism $\phi : M_b \to M_p$ such that $\phi^* g = \eta + h$, with $|h| \ll 1$, and where M_b is also a pseudo-Riemannian manifold. If the gravitational fields on M_p are weak then there will exist some diffeomorphism for which this is true. Let us now consider a one-parameter group of diffeomorphism $\psi_\epsilon : M_b \to M_b$, defined by the local flow of a vector field ξ given in local coordinates by $\xi = \xi^\mu \frac{d}{dx^\mu}$. If ϵ is sufficient small then $\phi \circ \psi_\epsilon$ will also obey $(\phi \circ \psi_\epsilon)^*\, g = \eta + h^\epsilon$, with a different perturbation h^ϵ such that $|h^\epsilon| \ll 1$. Expanding h^ϵ for ϵ infinitesimal one gets the known result [5, 6, 7, 8, 9]

$$h^\epsilon_{\mu\nu} = h_{\mu\nu} + 2\epsilon\partial_{(\mu}\xi_{\nu)} , \tag{1.17}$$

where the $(\mu\nu)$ stands for symmetrization on μ and ν. One can then check that such a transformation does not change the linearized Riemann tensor, thus leaving the physical spacetime unchanged.

From the ten initial degrees of freedom of the metric only six of them are physical, which means that the spurious degrees of freedom may be removed by fixing a gauge. There are several possible gauges, and we choose the harmonic gauge (also known as the Lorenz gauge, since its analogous to the Lorenz gauge in electromagnetism). Defining the trace-reversed perturbation $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$, this gauge corresponds to setting

$$\partial_\mu \bar{h}^{\mu\nu} = 0\,, \tag{1.18}$$

by choosing the vector field ξ^μ adequately. This perturbation is called trace-reversed because $\bar{h} = \eta^{\mu\nu}h_{\mu\nu} = -h$. Note that the original perturbation is not transverse in this gauge since $\partial_\mu h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\partial_\mu h = 0$. This gauge has the advantage of casting the Einstein equations in the simple form

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}\,. \tag{1.19}$$

Transverse traceless gauge

To study gravitational wave propagation and interaction one is interested in the wave equation outside the source, where $T^{\mu\nu} = 0$. In fact, in several situations one is only interested in the metric perturbation at large distances from the source, where some additional simplifications apply, as we shall see later on. Outside the sources the wave equation becomes

$$\Box h_{\mu\nu} = 0\,, \tag{1.20}$$

which means that the gauge fixing condition imposed in Eq. (1.18) fails to fix the gauge completely, as there is a residual freedom. This freedom corresponds to a further transformation, with ξ^μ satisfying $\Box\xi^\mu = 0$. We can fix this freedom by setting $h = 0$ and $h^{i0} = 0$ by choosing ξ^0 and ξ^i appropriately. Under these conditions the harmonic gauge condition (1.18) for $\mu = 0$ simplifies to $\partial_0 h^{00} = 0$, which means h^{00} is constant in time, corresponding to the static part of the gravitational interactions. Since the gravitational wave is time dependent, we can take $h_{00} = 0$, as far as gravitational waves are concerned. The only nonzero components are now the spatial ones h_{ij}. The harmonic gauge condition (1.18) for $\mu = i$ requires that $\partial_i h_{ij} = 0$. Thus we have reduced the previous six degrees of freedom to only two. Since $h_{\mu\nu}$ is traceless, there is no distinction between $h_{\mu\nu}$ and $\bar{h}_{\mu\nu}$. This gauge is known as the transverse traceless (TT) gauge, and it is denoted as $h^{\rm TT}_{\mu\nu}$. We now summarize the transverse traceless gauge conditions:

$$\begin{cases} h^{\rm TT}_{\mu 0} = 0 \\ h^{\rm TT} = 0 \\ \partial_i h^{\rm TT}_{ij} = 0 \end{cases} \tag{1.21}$$

Note that this gauge choice can not be applied inside the source, where $T^{\mu\nu} \neq 0$. Although we can still make the coordinate transformation described above, it can not be used to set to zero any component of $h_{\mu\nu}$ since it no longer satisfies $\Box h_{\mu\nu} = 0$. This is similar to what happens in electrodynamics with the Lorenz gauge outside the source.

One could now find solutions to the wave equation (1.20) in the TT gauge. The plane wave is such a solution. Considering a wave with 4-wave-vector $k^\mu = (\omega, \mathbf{k})$, in the TT gauge, the only non vanishing components of $h^{\rm TT}_{ij}$ are in a plane transverse to $\hat{\mathbf{k}} \equiv \mathbf{k}/|\mathbf{k}|$. Since gravitational radiation propagates at the speed of light, as can one can see from the wave equation where $\Box = -\partial_t^2 + \partial_i\partial^i$, we must have $k \cdot k = 0$, that is $\omega = |\mathbf{k}|$. This wave has then two polarizations, the "plus" h_+ and the "cross" $h_\times$ polarizations. These are the two polarizations expected for the graviton, since it is a massless spin–2 particle. We can define the two polarization tensors ϵ_I of a wave travelling in the $\hat{\mathbf{k}}$ direction by

$$\epsilon_\times = \frac{\sqrt{2}}{2}\left(\mathbf{u}\otimes\mathbf{v}+\mathbf{v}\otimes\mathbf{u}\right)\,, \quad \epsilon_+ = \frac{\sqrt{2}}{2}\left(\mathbf{u}\otimes\mathbf{u}-\mathbf{v}\otimes\mathbf{v}\right)\,, \tag{1.22}$$

where $I = \times, +$, and $\mathbf{u}, \mathbf{v}$ are unit vectors orthogonal to $\hat{\mathbf{k}}$. Note that $(\epsilon_I)_{ij}(\epsilon_{I'})^{ij} = \delta_{II'}$.

Consider now a wave travelling outside the sources from which it was emitted, in the harmonic gauge, but not yet in the TT gauge. It is possible to define a projector which allows us to find the metric perturbation in the TT gauge. To do so we begin by defining the tensor $P_{ij}(\mathbf{n})$ as

$$P_{ij}(\mathbf{n}) = \delta_{ij} - n_i n_j\,, \tag{1.23}$$

which is symmetric and transverse, meaning that $n^i P_{ij}(\mathbf{n}) = 0$. This tensor is also a projector since $P_{ij}(\mathbf{n})P_{jk}(\mathbf{n}) = P_{ik}(\mathbf{n})$, and its trace is $P = 2$. It projects vectors onto the surface with unit normal vector n, that is if V is a vector we have that $PV \cdot n = 0$. We choose n^i to point along the direction of propagation of the wave, so that P_{ij} projects onto a 2−sphere. This tensor can be used to build a projector onto the TT gauge. We accomplish this by defining $\Lambda_{ij,kl}$[3] as

$$\Lambda_{ij,kl}(\mathbf{n}) = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\,, \tag{1.24}$$

which is a transverse on all indices, traceless with respect to the first two and the last two indices, and is still a projector since $\Lambda_{ij,kl}\Lambda_{kl,mn} = \Lambda_{ij,mn}$. This tensor can be written explicitly in terms of $\mathbf{n}$ as

$$\Lambda_{ij,kl}(\mathbf{n}) = \delta_{ik}\delta jl - \frac{1}{2}\delta_{ij}\delta_{kl} - n_j n_l \delta_{ik} - n_i n_k \delta_{jl} + \frac{1}{2}n_k n_l \delta_{ij} + \frac{1}{2}n_i n_j \delta_{kl} + \frac{1}{2}n_i n_j n_k n_l\,, \tag{1.25}$$

which is symmetric for the change $ij \leftrightarrow kl$. If we have a gravitational wave in the harmonic gauge $h_{\mu\nu}$, which means it is a solution of $\Box h_{\mu\nu} = 0$ outside the source, then we can project this solution in the TT gauge with $\Lambda_{ij,kl}$ by

$$h^{\rm TT}_{ij} = \Lambda_{ij,kl}h_{kl}\,. \tag{1.26}$$

The $h^{\rm TT}_{ij}$ computed this way is a solution to $\Box h^{\rm TT}_{ij} = 0$ and it is transverse and traceless due to the properties of the projector. This method of computing the metric perturbation in the TT gauge is be very useful since we are interested in the field far from the source from which the waves far emitted, that is in vacuum. Since $h^{\rm TT}_{ij}$ is traceless we have $h^{\rm TT}_{ij} = \bar{h}^{\rm TT}_{ij} = \Lambda_{ij,kl}\bar{h}_{kl}$.

1.3 Generation of Gravitational Waves

In this Section we study the production of gravitational waves by sources. Since $T^{\mu\nu}$ no longer vanishes the TT gauge cannot be chosen, as explained above, and we must solve the wave equation (1.19). The solution of this equation is found by the convolution of the Green function and the source term $-16\pi T^{\mu\nu}$, plus an homogeneous solution which will be discarded. The metric perturbation is then

$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = -16\pi\int dt'\int d^{D-1}\mathbf{x}'\,T_{\mu\nu}(t',\mathbf{x}')G^{\rm ret}(t-t',\mathbf{x}-\mathbf{x}')\,, \tag{1.27}$$

[3]Note that the comma in the definition of this tensor is simply present to distinguish the first to indices from the last two, and it does not stand for partial differentiation, which is also commonly denoted by a comma. This notation was chosen in agreement with the standard notation in the literature [5, 8].

where $G^{\rm ret}$ denotes the retarded Green function for the d'Alembertian,

$$\eta^{\mu\nu}\partial_\mu\partial_\nu G(t-t',\mathbf{x}-\mathbf{x}') = \delta(t-t')\delta^3(\mathbf{x}-\mathbf{x}')\,. \tag{1.28}$$

We are only interested in the retarded Green function since it is the one which propagates signals forward in time. The retarded Green function is given by[12]

$$G^{\rm ret}(t-t',\mathbf{x}-\mathbf{x}') = -\frac{1}{4\pi}\frac{\delta(t-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}\Theta(t-t')\,. \tag{1.29}$$

Plugging this back in Eq. (1.27) we find

$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = 4\int d^3\mathbf{x}'\,\frac{1}{|\mathbf{x}-\mathbf{x}'|}T_{\mu\nu}(t_{\rm ret},\mathbf{x}')\,, \tag{1.30}$$

where the retarded time $t_{\rm ret} = t-|\mathbf{x}-\mathbf{x}'|$. This means that the sources at all the points in the past light cone of a given point contribute to the perturbation in the gravitational field at that point.
This is the exact solution to the wave equation in linearized gravity. However in radiation problems one is only interested in the field at large distances from the sources, that is $r \equiv |\mathbf{x}| \gg R_s$ and also $r \gg \lambda$ and $r \gg R_s^2/\lambda$, where R_s is the source's dimension and λ the wavelength of the wave. This is defined as the wave zone, and in such conditions we can make the following approximation

$$|\mathbf{x}-\mathbf{x}'| \simeq r-\mathbf{x}'\cdot\mathbf{n}\,,\quad \mathbf{n} = \mathbf{x}/r\,. \tag{1.31}$$

The leading term in Eq. (1.30) is then

$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = \frac{4}{r}\int d^3\mathbf{x}'\,T_{\mu\nu}(t_{\rm ret},\mathbf{x}')\,, \tag{1.32}$$

and $t_{\rm ret}$ can be approximated by $t-(r-\mathbf{x}'\cdot\mathbf{n})$.

Now we can proceed to Fourier-analyse the metric perturbation. The conventions for the Fourier transforms are defined in Sec. 1.1. If we express the energy-momentum tensor in terms of its Fourier transform

$$T_{\mu\nu}(t,\mathbf{x}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3\mathbf{k}\int_{-\infty}^{+\infty} d\omega\,\tilde{T}_{\mu\nu}(\omega,\mathbf{k})e^{-i\omega t+i\mathbf{k}\cdot\mathbf{x}}\,, \tag{1.33}$$

and replace in Eq. (1.32), approximating $t_{\rm ret}$, and performing the integration in $\mathbf{x}'$ and $\mathbf{k}$, the trace-reversed metric perturbation becomes

$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = \frac{4}{r}\int d\omega\,\tilde{T}_{\mu\nu}(\omega,\omega\mathbf{n})e^{i\omega(t-r)}\,. \tag{1.34}$$

Quadrupole approximation

So far we have made a weak field approximation, which is valid whenever the fields are sufficiently weak to assume the background to be flat. Note that we can always choose r large enough so that the assumption that the fields are measured in the wave zone is valid, and since that is the quantity one is interested in, for radiation problems, we shall continue to make that assumption. The weak field approximation does not, however, necessarily imply small velocities. For a system held together by gravitational forces, weak fields imply that the typical velocities inside the source are small, whereas for systems with dynamics determined by nongravitational forces the weak field approximation may be taken independently of the velocities. This means that for systems whose dynamics are determined by nongravitational forces we can consider weak fields and arbitrary velocities. We now consider a further approximation, by assuming that the typical velocities in a given system are small, that is $v \ll 1$.

This is the quadrupole approximation of the metric perturbation, which, as we shall see, provides a much simpler way to compute it, valid as long as the velocities are small, and it was first derived by Einstein [13]. For several systems it would become quite complicated to compute the full metric perturbation, and the fact that for this approximation one only has to consider the 00 component of the energy-momentum tensor simplifies matters significantly. This will become clear with the example which is considered in Sec. 1.4.

Let us assume that the typical velocities inside the source of gravitational radiation are small $v \ll 1$. If R_s denotes the source's size, and ω_s the typical frequency of the motion inside the source, one has $v \sim R_s\omega_s$. This approximation implies the following assumption, regarding the emitted radiation's reduced wavelength $\lambdabar$, $\lambdabar \gg R_s$, where we have used $\lambdabar = 1/\omega \sim \omega_s$, and where ω denotes the radiation frequency, which we assume to be of the same order of magnitude as the typical frequencies inside the source.

Once again, we Fourier-analyse the energy-momentum tensor, Eq. (1.33), and replace this expression in the metric perturbation (1.32). Since the field is being computed in the wave zone $t_{\rm ret}$ is approximated by $t - (r - \mathbf{x}' \cdot \mathbf{n})$, and we can expand the exponential for $\omega\mathbf{x}' \cdot \mathbf{n} \lesssim R_s\omega_s \ll 1$[4]. We get

$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = \frac{4}{r}\int d^3\mathbf{x}' \int_{-\infty}^{+\infty} \frac{d^3\mathbf{k}}{(2\pi)^3} \int_{-\infty}^{+\infty} d\omega \tilde{T}_{\mu\nu}(\omega,\mathbf{k}) e^{-i\omega(t-r)-i\omega\mathbf{k}\cdot\mathbf{x}'} \left(1 - i\omega\mathbf{x}' \cdot \mathbf{n} + \dots\right) , \tag{1.35}$$

thus by integrating in ω and $\mathbf{k}$ we have,

$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = \frac{4}{r}\int d^3\mathbf{x}' \left(T_{\mu\nu}(t-r,\mathbf{x}') + \mathbf{x}' \cdot \mathbf{n}\, \partial_t T_{\mu\nu}(t-r,\mathbf{k}') + \dots\right) . \tag{1.36}$$

The quadrupole approximation is obtained considering only the leading term in the expansion, while higher multipoles come from higher order terms. In this Section we focus only on the leading term, and we leave the next-to-leading order contribution for Sec. 2.2.

Keeping in mind that we have fixed the gauge to be the harmonic gauge, the only metric components we need are the purely space-like ones. Indeed, if we have $\bar{h}^{ij}$ we can compute $\bar{h}^{0j}$ using the harmonic gauge condition (1.18), and also $\bar{h}^{00}$ from $\bar{h}^{0j}$ using the same condition. The conservation of the energy-momentum tensor in turn allows us to compute the metric knowing only its 00 component. Integrating by parts in reverse $\int d^3\mathbf{x}' T_{ij}(t-r,\mathbf{x}')$, and imposing that $\partial_\mu T^{\mu\nu} = 0$ we have that

$$\int d^3\mathbf{x}' T_{ij}(t-r,\mathbf{x}') = \frac{1}{2}\partial_t^2 \int d^3\mathbf{x}' x'^i x'^j T_{00}(t-r,\mathbf{x}') , \tag{1.37}$$

where we have used the fact that the source is isolated, thus the integration of perfect divergencies vanishes, and that the left-hand side of Eq. (1.36) must be symmetric in $\mu\nu$ since the metric is.
It is conventional to define the quadrupole moment tensor of the energy density T^{00} by

$$D^{ij}(t) = \int d^{D-1}\mathbf{x}\, x^i\, x^j\, T^{00}(t,\mathbf{x}) . \tag{1.38}$$

Finally one obtains the quadrupole formula

$$\bar{h}_{ij}(t,\mathbf{x}) = \frac{2}{r}\partial_t^2 D_{ij}(t-r) . \tag{1.39}$$

Note that outside the source we can always use the projectors $\Lambda_{ij,kl}$ to project the metric in the TT gauge.

[4] The integration is over $\mathbf{x}'$ and the only contributions will arise from inside the source, where $T^{\mu\nu}$ is nonvanishing, therefore $|\mathbf{x}'|$ will be smaller or equal to the source's dimension R_s.

As pointed out above the same procedure can be used to obtain a systematic expansion of the metric perturbation in order to get higher multipolar terms (this expansion is treated with great detail in [8]). In the next Chapter we consider the next-to-leading order term in this expansion, the octopole term. Instead of expanding Eq. (1.32) we expand a totally equivalent expression, which was first derived by Press in [3]. We deduce this formula in higher dimensional spacetimes, and take the small velocity expansion to obtain the quadrupole-octopole formula, which will be used to study two different systems.

1.4 Energy and Momentum of Gravitational Waves

A natural question regarding gravitational wave emission is the energy and momentum they carry. Since gravitational waves carry energy they should also contribute to the spacetime's curvature, which is what we consider in this Section. In the linearized approach we treated the gravitational waves as the perturbation to a flat background metric, but to compute the gravitational waves' contribution to the curvature we have to go beyond this linearized approximation. The linearized approximation is equivalent to a field theory with a massless spin-2 particle (the graviton) propagating on a fixed background metric. This particle corresponds to the metric perturbation, so similarly to what is done for other field theories we will try to derive an energy-momentum tensor for the perturbation $h_{\mu\nu}$.

In order to study the energy carried by gravitational waves we will go back to the expansion about flat space, and consider the propagation of waves in vacuum. Let us take the following expansion of the spacetime metric about flat space, where we have considered not only the first order term $\epsilon h^{(1)}_{\mu\nu}$, which was formerly denoted just by $h^{\mu\nu}$, but also next order term $\epsilon^2 h^{(2)}_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h^{(1)}_{\mu\nu} + \epsilon^2 h^{(2)}_{\mu\nu}\,, \tag{1.40}$$

where we have that $\epsilon \ll 1$ is a small adimensional parameter, and $\epsilon^2 h^{(2)}_{\mu\nu}$ is of the same order as $\left(\epsilon h^{(1)}_{\mu\nu}\right)^2$. We will now consider the Einstein equations order by order in ϵ. Let $R^{(i)}_{\mu\nu}$, $i = 0, 1, 2$ denote the terms in the Ricci tensor $R_{\mu\nu}$ independent, linear and quadratic on the metric respectively. The zeroth order Einstein equations give $R^{(0)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(0)} = 0$, which is always true since our background $\eta_{\mu\nu}$ is already solution to Einstein equations. The first order terms give $R^{(1)}_{\mu\nu}[\epsilon h^{(1)}] - \frac{1}{2}\eta_{\mu\nu}R^{(1)}[\epsilon h^{(1)}] = 0$, which was the equation found previously, and the one from which the metric perturbation is obtained. Notice that the term in $\epsilon h^{(1)}_{\mu\nu}R^{(0)}$ vanishes because $R^{(0)} = 0$ from the previous order equation. Let us turn now to the second order terms, which we have neglected in the previous Sections, they yield the following equations

$$R^{(1)}_{\mu\nu}[\epsilon^2 h^{(2)}] + R^{(2)}_{\mu\nu}[\epsilon h^{(1)}] - \frac{1}{2}\eta_{\mu\nu}R^{(1)}[\epsilon^2 h^{(2)}] - \frac{1}{2}\eta_{\mu\nu}R^{(2)}[\epsilon h^{(1)}] = 0\,, \tag{1.41}$$

where we have to consider two contributions, the terms from the Ricci tensor quadratic in the metric perturbation, which are denoted by $R^{(2)}_{\mu\nu}[\epsilon h^{(1)}]$ and are computed for the metric term of order ϵ; and the Ricci tensor terms linear in the metric perturbation, which will have a second order contribution from the metric perturbation in second order $\epsilon^2 h^{(2)}_{\mu\nu}$, and are denoted by $R^{(1)}_{\mu\nu}[\epsilon^2 h^{(2)}]$. Notice that, once again, terms like $\epsilon^2 h^{(2)}_{\mu\nu}R^{(0)}$ and $\epsilon h^{(1)}_{\mu\nu}R^{(1)}[h^{(1)}]$ vanish by the previous equations. We can now re-write this equation in the following way

$$R^{(1)}_{\mu\nu}[\epsilon^2 h^{(2)}] - \frac{1}{2}\eta_{\mu\nu}R^{(1)}[\epsilon^2 h^{(2)}] = 8\pi t_{\mu\nu}\,, \quad t_{\mu\nu} = -\frac{1}{8\pi}\left(R^{(2)}_{\mu\nu}[\epsilon h^{(1)}] - \frac{1}{2}\eta_{\mu\nu}R^{(2)}[\epsilon h^{(1)}]\right)\,, \tag{1.42}$$

in which the right-hand side will be interpreted as the first order perturbation energy-momentum tensor. This gravitational energy-momentum tensor will be the one used to obtain the energy carried by gravitational waves. The Bianchi identities also imply, since we are far from the sources, that this tensor is conserved $\partial_\mu t^{\mu\nu} = 0$

There is one major problem with this identification, this gravitational energy-momentum tensor is not invariant under diffeomorphisms. We can solve this problem by always considering $t_{\mu\nu}$ to be an average over several wavelengths.

Since we are considering Einstein equations in vacuum we can go to the TT gauge to compute the gravitational energy-momentum tensor. Let $\langle \cdots \rangle_{av}$ denote the average over several wavelengths. In this gauge we have

$$t_{\mu\nu} = \frac{1}{32\pi} \langle \partial_\mu h^{\mathrm{TT}}_{\rho\sigma} \partial_\nu h^{\mathrm{TT}\rho\sigma} \rangle_{av} \,, \tag{1.43}$$

where we have once again denoted the first order perturbation by $h_{\mu\nu}$ instead of $\epsilon h^{(1)}_{\mu\nu}$. This quantity is known as the Isaacson energy-momentum tensor [14]. Now that we have an expression for the gravitational energy-momentum tensor we can proceed to compute the radiated energy and linear momentum, which is carried out in the following Sections.

So far we have taken the source of the wave equation to be simply $T^{\mu\nu}$, that is, the matter energy-momentum tensor. However in some cases one may need to consider the contribution from the gravitational energy-momentum tensor as well. The conditions under which this term has to be also considered were studied in Ref. [15]. Under these circumstances we have to take the source term to be an effective energy-momentum tensor $\mathcal{T}^{\mu\nu} = T^{\mu\nu} + t^{\mu\nu}$, which is conserved as a consequence of the Bianchi identities, i.e. $\partial_\nu \mathcal{T}^{\mu\nu} = 0$.

Radiated energy

The total gravitational energy inside a given surface Σ is

$$E_V = \int_V d^3\mathbf{x}\, t_{00} \,, \tag{1.44}$$

where V is the volume bounded by the surface Σ. Using the conservation of the gravitational energy-momentum tensor, we have $\int_V d^3\mathbf{x} \partial_\mu t^{\mu\nu} = 0$, so

$$\frac{dE_V}{dt} = -\int_V d^3\mathbf{x}\, \partial_i t^{i0} = -\int_\Sigma d\Sigma\, n_i t^{0i} \,, \tag{1.45}$$

where n is an out-pointing normal unit vector to Σ, and $d\Sigma$ is the surface element. If we consider a $2-$sphere at spatial infinity, we have that the energy radiated per unit of time through the sphere is

$$\frac{dE_V}{dt} = -\int_{S^2_\infty} d\Omega\, t^{0r} r^2 \,, \tag{1.46}$$

since $d\Sigma = r^2 d\Omega$ and $\mathbf{n} = \mathbf{e}_r$. From the previous Section we have that

$$t^{0r} = \frac{1}{32\pi} \langle \partial^0 h^{\mathrm{TT}}_{ij} \partial_r h^{\mathrm{TT}}_{ij} \rangle \,. \tag{1.47}$$

At large distances from the source we have that $t^{0r} = t^{00}$, since at large distances the metric is given by Eq. (1.32), where the second part depends only on $t_{\mathrm{ret}} = t - r$, which means $\partial_r h^{\mathrm{TT}}_{ij} = -\partial_0 h^{\mathrm{TT}}_{ij} + O(1/r^2)$. This means that Eq. (1.46) can be re-written as $\frac{dE_V}{dt} = -\int_{S^2_\infty} d\Omega\, t^{00} r^2$. Note that the energy that leaves the sphere is negative, so the gravitational waves carry a energy flux given by

$$\frac{d^2E}{dtd\Omega} = \frac{r^2}{32\pi} \left\langle \dot{h}^{\mathrm{TT}}_{jk} \dot{h}^{\mathrm{TT}}_{jk} \right\rangle_{av} \,, \tag{1.48}$$

where the dot over h denotes a time derivative. When computing the total energy, that is, when one integrates in time, we can perform the integration before taking the average over several wavelengths $\langle \cdots \rangle_{av}$, and then we would have the temporal average of a constant, and the average may be omitted.

We can now compute the energy carried by gravitational waves for a given metric perturbation, for that we just have to project the perturbation onto the TT gauge as seen in the previous Sections. However there is another useful expression one can derive which allows us to compute the radiated energy, per unit of frequency, directly from the Fourier transforms of the source's energy-momentum tensor. In the last part of this Section we derive this expression. Going back to Eq. (1.34) we see that at large distances from the source it can be approximated by a plane wave,

$$\bar{h}_{\mu\nu} = \int d\omega \, \epsilon_{\mu\nu}(\omega, \mathbf{k}) e^{ik\cdot x} \,, \tag{1.49}$$

where the wave vector is $(k)^\mu = (\omega, \omega\mathbf{n})$ and the polarization tensors are

$$\epsilon_{\mu\nu}(\omega, \mathbf{k}) = \frac{4}{r} \tilde{T}_{\mu\nu}(\omega, \mathbf{k}) \,. \tag{1.50}$$

Computing the gravitational energy-momentum tensor for such a plane wave, imposing only the harmonic gauge conditions, one finds,

$$t^{\mu\nu} = \frac{k^\mu k^\nu}{r^2 \pi} \left(\tilde{T}^{\lambda\rho} \tilde{T}_{\lambda\rho} - \frac{1}{2} |\tilde{T}^\lambda{}_\lambda|^2 \right) . \tag{1.51}$$

Plugging this back in Eq. (1.45), and keeping in mind that the energy carried by the gravitational waves is $\frac{d^2E}{dtd\Omega} = -\frac{d^2E_V}{dtd\Omega}$, one gets[5]

$$\frac{d^2E}{d\omega d\Omega} = 2\omega^2 \left(\tilde{T}^{\mu\nu}(\omega, \mathbf{k}) \tilde{T}^*_{\mu\nu}(\omega, \mathbf{k}) - \frac{1}{2} \left| \tilde{T}^\lambda{}_\lambda(\omega, \mathbf{k}) \right|^2 \right) , \tag{1.52}$$

where the star stands for complex conjugation and the direction in which the energy is radiated is $\hat{\mathbf{k}} = \mathbf{k}/\omega$. The energy can also be expressed in terms of the purely space-like components of $\tilde{T}^{\mu\nu}$. The conservation equation for $T^{\mu\nu}$ implies that its Fourier transforms obey the following relations $k_\mu \tilde{T}^{\mu\nu}(\omega, \mathbf{k}) = 0$, so it is possible to write $\tilde{T}^{00}$ and $\tilde{T}^{0i}$ in terms of $\tilde{T}ij$

$$\begin{aligned} \tilde{T}_{00}(\omega, \mathbf{k}) &= \hat{k}^i \hat{k}^j \tilde{T}_{ij}(\omega, \mathbf{k}) \,, \\ \tilde{T}_{0i}(\omega, \mathbf{k}) &= -\hat{k}^j \tilde{T}_{ij}(\omega, \mathbf{k}) \,. \end{aligned} \tag{1.53}$$

With these identities at hand, Eq. (1.52) can be written as[6]

$$\frac{d^2E}{d\omega d\Omega} = 2\omega^2 \Lambda_{ij,lm}(\hat{k}) \tilde{T}^{*ij}(\omega, \mathbf{k}) \tilde{T}^{lm}(\omega, \mathbf{k}) \,, \tag{1.54}$$

where $\Lambda_{ij,lm}(\hat{k})$ is the projector onto the TT gauge Eq. (1.25). If one considers a system of freely moving point particles, with 4-momenta $(p_j)^\mu = (E_j, \gamma_j m_j \mathbf{v}_j)$, which suffer a sudden collision at $t = 0$, changing their velocities abruptly to $\mathbf{v}'$, the energy-momentum tensor is

$$T^{\mu\nu}(t, \mathbf{x}) = \sum_j \frac{p^\mu_j p^\nu_j}{E_j} \delta^3(\mathbf{x} - \mathbf{v}t)\Theta(-t) + \sum_j \frac{p'^\mu_j p'^\nu_j}{E'_j} \delta^3(\mathbf{x} - \mathbf{v}'t)\Theta(t) \,, \tag{1.55}$$

where the primes denote final states, and the sums run over the particles labelled by j. Taking the Fourier transform of this equation and plugging back in Eq. (1.52) we find the following radiated energy

$$\frac{d^2E}{d\omega d\Omega} = \frac{\omega^2}{2\pi^2} \sum_{N,M} \eta_N \eta_M \frac{(p^N \cdot p^M)^2 - \frac{1}{2} m_N^2 m_M^2}{k \cdot p^N k \cdot p^M} \,, \tag{1.56}$$

where η_N is equal to ± 1 for particles final and initial states respectively, and the sums run over all the particles both in the initial and in the final states.

[5]Note that if the conventions for the Fourier transforms had been different this expression would be changed (see Sec. 1.1), if we had defined the Fourier transform with respect to the frequency with $d\omega/(2\pi)$ our energy would differ from Eq. (1.52) by factor $1/(4\pi^2)$.

[6]This is what we would have obtained if we had projected Eq. (1.34) in the TT gauge and plugged it in Eq. (1.48).

Radiated energy in the quadrupole approximation

For slow moving sources we can use the quadrupole approximation derived in Sec. 1.3 to estimate the total radiated energy. Projecting the metric perturbation Eq. (1.39) onto the TT gauge and plugging in Eq. (1.48) we can integrate over the solid angle $d\Omega$ using the integrals from Appendix A.1. Finally, the radiated energy per unit of time in the quadrupole approximation is given by

$$\frac{dE}{dt} = \frac{1}{5}\langle \partial_t^3 D_{ij}(t)\partial_t^3 D^{ij}(t) - \frac{1}{3}\left|\partial_t^3 D_{ij}(t)\right|^2\rangle_{av} \,. \tag{1.57}$$

We can also consider the Fourier transform to get the frequency spectrum, and we get[7]

$$\frac{dE}{d\omega} = \frac{4\pi\omega^6}{5}\left(\tilde{D}^{ij}(\omega)D^*_{ij}(\omega) - \frac{1}{3}\left|\tilde{D}_{ij}(\omega)\right|^2\right) \,. \tag{1.58}$$

Example: rotating body in the quadrupole approximation

As an example of the quadrupole approximation we compute the energy radiated by a (slowly) rotating body. Let us consider a body rigidly rotating about one of its principle axis. Without loss of generality we consider the body is rotating about the z–axis. If the body has a density $\rho(\mathbf{x}')$, where the prime denotes a coordinate system rotating with the body, we have that the 00 component of the energy-momentum tensor is, for a slow rotating body, $T^{00}(t,\mathbf{x}) = \rho(\mathbf{x}')$. If the body, and the coordinate system $\{x'\}$ is rotating with frequency Ω we have that the only nonvanishing components of the quadrupole moment tensor are (Eq. (1.38))

$$\begin{aligned}
D_{11}(t) &= \frac{I_1 - I_2}{2}(1 + \cos 2\Omega t) \\
D_{22}(t) &= \frac{I_1 - I_2}{2}(1 - \cos 2\Omega t) \\
D_{33}(t) &= I_3 \\
D_{12}(t) &= D_{21}(t) = \frac{I_1 - I_2}{2}\sin 2\Omega t \,,
\end{aligned} \tag{1.59}$$

where I_i are the body's principal moments of inertia. The trace-reversed metric perturbation is given by Eq. (1.39)

$$\bar{h}_{11} = -\bar{h}_{22} = -\frac{4\Omega^2}{r}(I_1 - I_2)\cos 2\Omega t_{\text{ret}} \tag{1.60a}$$

$$\bar{h}_{12} = -\frac{4\Omega^2}{r}(I_1 - I_2)\sin 2\Omega t_{\text{ret}} \,. \tag{1.60b}$$

We can now project the metric onto the TT gauge, considering the observation direction to be $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, using Eq. (1.26). As discussed above, the "plus" and "cross" metric perturbations correspond to the metric components h'^{TT}_{11} and h'^{TT}_{12} respectively, in an orthonormal frame $\{X_1, X_2, X_3\}$, where $X_3 = \mathbf{n}$. We take this frame to be one such that when $\mathbf{n} = \mathbf{e}_z$ we have $X_1 = \mathbf{e}_x$ and $X_2 = \mathbf{e}_y$. By performing a rotation to the $\{X_1, X_2, X_3\}$ frame, we find

$$h_+ = \frac{2(I_1 - I_2)\Omega^2}{r}(1 + \cos^2\theta)\cos(2\phi - 2\Omega t_{\text{ret}}) \,, \quad h_\times = -\frac{4(I_1 - I_2)\Omega^2}{r}\cos\theta\sin(2\phi - 2\Omega t_{\text{ret}}) \,. \tag{1.61}$$

Performing the Fourier transforms we see that the energy is radiated only at $\omega = 2\Omega$, so we have

$$\frac{dE}{dt} = \frac{32}{5}\Omega^6(I_1 - I_2)^2 \,. \tag{1.62}$$

This emission at twice the rotating frequency for a rotating body is recovered in Secs. 2.2 and 3.2 for two particles in circular orbit. We also see that if the body has circular symmetry around the rotation axis z we have $I_1 = I_2$, and it does not radiate, a property that holds even when the quadrupole approximation is not valid [5].

[7] Once again different conventions in the definition of the Fourier transform must be taken into account in order to compare this expression with other references with different conventions, such as [8].

Radiated momentum

Gravitational waves carry not only energy, but also linear and angular momentum. Momentum transport by gravitational waves was first considered in [16], and has been widely studied. An interesting consequence of momentum emission through gravitational waves is the recoil effect in the source due to the global conservation of momentum [16, 17, 18]. As we shall see, the quadrupole approximation is insufficient to compute the radiated momentum, since it appears in the quadrupole-octopole cross terms at the lowest order as found.

To compute the radiated momentum we proceed in a similar way. The momentum emitted through gravitational waves in the direction j inside a volume V at large distances from the source is

$$P_V^j = \int_V d^3\mathbf{x}\, t^{0j}\,, \tag{1.63}$$

then from the conservation of the gravitational energy-momentum tensor, we have that

$$\frac{dP_V^j}{dt} = -\int_V d^3\mathbf{x}\,\partial_i t^{ij} = -\int_\Sigma d\Sigma\, n_i t^{ji} = -\int_\Sigma d\Sigma\, t^{j0}\,. \tag{1.64}$$

Considering once again a 2−sphere, the momentum flux carried by gravitational waves is,

$$\frac{d^2P^j}{dtd\Omega} = -\frac{r^2}{32\pi}\langle \dot{h}_{il}^{\mathrm{TT}}\partial^j h^{\mathrm{TT}\,il}\rangle_{av}\,. \tag{1.65}$$

On the other hand, from Eq. (1.64) we have

$$\frac{d^2P^j}{dtd\Omega} = r^2 n_i t^{ji} = n^j\frac{d^2E}{dtd\Omega}\,. \tag{1.66}$$

Therefore radiated momentum is given by the integration of the energy $\frac{d^2E}{dtd\Omega}$ over a two-sphere at infinity, S_∞, centred on the coordinate origin,

$$\frac{dP^i}{dt} = \int_{S_\infty} d\Omega\,\frac{d^2E}{dtd\Omega}n^i\,, \tag{1.67}$$

where n^i is a unit radial vector on S_∞. If we were to consider the energy radiated at quadrupole order, in which the only angular dependence is on $\Lambda_{ij,kl}$ the integral in (1.67) would vanish, due to parity.

Chapter 2

Multipolar Expansion of the Metric Perturbation

The exact solution of Eq. (1.30) proves to be hard to compute in several situations, while the quadrupole approximation discussed in the previous chapter (Sec. 1.3) has the advantage of greatly simplifying the calculations involved, for nonrelativistic systems. As mention previously, this expansion can be taken up to higher orders in a systematic fashion. In this chapter we derive a multipolar expansion for the metric perturbation in higher dimensional spacetimes, with an even number of dimensions. This expression is obtained expanding a formula for the metric perturbation (equivalent to (1.32))first found by Press [3], which we now generalize to higher dimensional spacetimes. This expansion assumes that the source is small and that the velocities are low, so naturally the first term in this expansion is the quadrupolar approximation (see Sec. 1.3). This first term allows one to compute the radiated energy in a given system, and although it is a small velocity approximation it provides a quite good approximation to some processes, even when the velocities involved are not always low. The second term in the multipolar expansion is the octopole term. Clearly this term provides a correction to the radiated energy computed only using the quadrupolar approximation, but it also allows us to estimate the total radiated momentum in this process. Higher order terms are obtained in a similar way, but we are only concerned with the first two.
In the last Sections of this Chapter we use the quadrupole-octopole formula to compute the radiated energy and momentum for a particle falling along a radial geodesic into a $D-$dimensional Schwarzschild-Tangherlini black hole, and for a particle in circular orbit.

2.1 Press Formula in Higher Dimensional Spacetimes

In 1977 Press [3] derived a formula which is an exact replacement for Eq. (1.32), which means that it is valid whenever one is computing the field at large distances from the source. It is similar to the quadrupole approximation discussed in Chapter 1, but it does not assume slow motion or small sources. In fact, it includes not only the quadrupole term but also the octopole and all higher multipoles, which can be obtained expanding the equation for small sources and low velocities. In the following Section we proceed in a way similar to Press in Ref. [3] to derive this formula in higher (even) dimensional spacetimes.

As seen in Chapter 1, when gravitational field are weak, one can expand the metric around a background metric, which we considered to be Minkowski flat. This then leads to a wave equation to the metric perturbation $h_{\mu\nu}$, with a source term which is the energy-momentum tensor, Eq. (1.19). Here we shall consider the generalization of the linearized approach to higher dimensional spacetimes, in order to derive Press's formula. The study the linearized Einstein equations in $D-$dimensions was done by Cardoso *et al.* in Ref. [10]. This proceeding is briefly described here. The metric is expanded as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $|h_{\mu\nu}| \ll 1$. As discussed in Sec. 1.2 there remains a gauge freedom, which is fixed choosing the harmonic gauge, $\partial_\mu h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\partial_\mu h = 0$. If one expands the Einstein field equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$, keeping only the lowest order terms, one finds, in this particular gauge

$$\Box h_{\mu\nu} = -16\pi S_{\mu\nu}\,, \tag{2.1}$$

where the wave equation was cast in this simple form by the definition of $S_{\mu\nu}$, in $D-$dimensions as,

$$S_{\mu\nu} = T_{\mu\nu} - \frac{1}{D-2}\,\eta_{\mu\nu}\,T^{\alpha}{}_{\alpha}\,, \tag{2.2}$$

instead of writing the wave equation in terms of the trace-reversed perturbation ($\bar{h}^{\mu\nu}$) as in Eq. (1.19).

Note that the energy-momentum tensor is taken to be just that of matter, therefore the Bianchi identities imply that $\partial_\nu T^{\mu\nu} = 0$. The general solution to this equation is determined by the convolution of the Green function $G(\mathbf{x}-\mathbf{x}', t-t')$ and the source term $-16\pi S_{\mu\nu}$. The Green function satisfies

$$\eta^{\mu\nu}\partial_\mu\partial_\nu G(t-t', \mathbf{x}-\mathbf{x}') = \delta(t-t')\delta(\mathbf{x}-\mathbf{x}')\,, \tag{2.3}$$

and $h_{\mu\nu}$ is

$$h_{\mu\nu}(t,\mathbf{x}) = -16\pi\int dt'\int d^{D-1}\mathbf{x}' S_{\mu\nu}(t',\mathbf{x}')G(t-t',\mathbf{x}-\mathbf{x}') + \text{homogeneous solutions}\,, \tag{2.4}$$

The retarded Green function (which is the one that propagates signals into the future) is, for even D

$$G^{\rm ret}(t,\mathbf{x}) = -\frac{\Theta(t)}{4\pi}\left[-\frac{\partial}{2\pi r\partial r}\right]^{(D-4)/2}\left[\frac{\delta(t-r)}{r}\right]. \tag{2.5}$$

The authors of [10] also compute the Green function for odd dimensions, however its analytical structure makes it hard to study gravitational waves in these spacetimes. The difference between even and odd dimensions is that for odd dimensions the Green function no longer depends solely on delta functions and its derivatives. As in [10] we restrict our study to even dimensional spacetimes only.

The general solution to Eq. (2.1), discarding the homogeneous solution and re-introducing the trace-reversed perturbation, is given by

$$\bar{h}^{\mu\nu}(t,\mathbf{x}) = -16\pi\int dt'\int d^{D-1}\mathbf{x}'\,T^{\mu\nu}(t',\mathbf{x}')G^{\rm ret}(t-t',\mathbf{x}-\mathbf{x}')\,, \tag{2.6}$$

where $\bar{h}^{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\,h^{\alpha}{}_{\alpha}$. In radiation problems one is only interested on the metric perturbation far from the source, that is in the wave zone, as discussed in Sec. 1.3. This means we can make the following approximation, assuming that the field is being computed at sufficiently large distances from the source and also at a distance much larger than the source's dimensions,

$$\bar{h}^{\mu\nu}(t,\mathbf{x}) = 8\pi\frac{1}{(2\pi r)^{(D-2)/2}}\partial_t^{(\frac{D-4}{2})}\left[\int d^{D-1}\mathbf{x}' T^{\mu\nu}(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}')\right]. \tag{2.7}$$

As shown in Chapter 1, all information about the outgoing gravitational radiation, outside the source, is contained in the spatial components $\bar{h}^{ij}$, in fact in only the traceless projection of $\bar{h}^{ij}$ (TT gauge).

Fourier-analysing the energy-momentum tensor, we have

$$T^{\mu\nu}(t,\mathbf{x}) = \int_{-\infty}^{\infty} d\omega \tilde{T}^{\mu\nu}(\omega,\mathbf{x})e^{-i\omega t}\,, \tag{2.8}$$

and replacing this in Eq. (2.7) we find

$$\bar{h}^{ij}(t,\mathbf{x}) = 8\pi\frac{1}{(2\pi r)^{(D-2)/2}}\partial_t^{(\frac{D-4}{2})}\left[\int d^{D-1}\mathbf{x}'\int_{-\infty}^{\infty} d\omega\tilde{T}^{ij}(\omega,\mathbf{x}')e^{-i\omega(t-|\mathbf{x}-\mathbf{x}'|)}\right]. \tag{2.9}$$

Since we are interested in the metric perturbation far from the source, meaning large $r = |\mathbf{x}|$, we can expand $|\mathbf{x} - \mathbf{x}'|$ for large r,

$$\bar{h}^{ij}(t, \mathbf{x}) = 8\pi \frac{1}{(2\pi r)^{(D-2)/2}} \partial_t^{(\frac{D-4}{2})} \left[\int_{-\infty}^{\infty} d\omega e^{i\omega r - i\omega t} \int d^{D-1}\mathbf{x}' T^{ij}(\omega, \mathbf{x}') e^{-i\omega \mathbf{n}\cdot\mathbf{x}'} \right], \tag{2.10}$$

where $\mathbf{n} = \mathbf{x}/r$ as in Sec. 1.3. Using Eq. (1.16) one can show that

$$\begin{aligned} \partial_l \partial_m \left(T^{lm} x^i x^j e^{-i\omega \mathbf{n}\cdot\mathbf{x}} \right) &= x^i x^j e^{-i\omega \mathbf{n}\cdot\mathbf{x}} \left(-\omega^2 T^{00} + 2\omega^2 n_m T^{0m} - \omega^2 n_l n_m T^{lm} \right) \\ &+ 2\partial_l \left(\partial_m \left(x^i x^j \right) T^{lm} e^{-i\omega \mathbf{n}\cdot\mathbf{x}} \right) - 2T^{ij} e^{-i\omega \mathbf{n}\cdot\mathbf{x}} . \end{aligned} \tag{2.11}$$

Substituting this in Eq. (2.10) yields

$$\bar{h}^{ij}(t, \mathbf{x}) = \frac{4\pi}{(2\pi r)^{(D-2)/2}} \partial_t^{(\frac{D-4}{2})} \left[\int_{-\infty}^{\infty} d\omega (-\omega^2) e^{i\omega r - i\omega t} \int d^{D-1}\mathbf{x}' x'^i x'^j e^{-i\omega \mathbf{n}\cdot\mathbf{x}'} \left(\tilde{T}^{00} - 2n_m \tilde{T}^{0m} + n_l n_m \tilde{T}^{lm} \right) \right], \tag{2.12}$$

where the surface terms arising from the integration of perfect divergences vanish. Now the integration in ω yields the Press formula for general (even) D-dimensional spacetimes

$$\bar{h}^{ij}(t, \mathbf{x}) = 4\pi \frac{1}{(2\pi r)^{(D-2)/2}} \partial_t^{(\frac{D-4}{2})} \left[\frac{d^2}{dt^2} \int d^{D-1}\mathbf{x}' x'^i x'^j \left(T^{00} - 2n_m T^{0m} + n_l n_m T^{lm} \right) \right], \tag{2.13}$$

where $T^{\mu\nu}$ under an integral means $T^{\mu\nu}(t_{\rm ret}, \mathbf{x}')$, which means that the energy-momentum tensor and its time derivatives must be evaluated at a retarded time $t_{\rm ret} = t - |\mathbf{x} - \mathbf{x}'|$ for each $\mathbf{x}'$ before integrating over $d^{D-1}\mathbf{x}'$. In four dimensions we get Eq. (8) of Ref. [3]

$$\bar{h}^{ij}(t, \mathbf{x}) = \frac{2}{r} \left[\frac{d^2}{dt^2} \int d^3\mathbf{x}' x'^i x'^j \left(T^{00} - 2n_m T^{0m} + n_l n_m T^{lm} \right) \right]. \tag{2.14}$$

This equation reduces to the quadrupole formula, if one considers small sources and low velocities,

$$\bar{h}^{ij}(t, \mathbf{x}) = 4\pi \frac{1}{(2\pi r)^{(D-2)/2}} \partial_t^{(\frac{D-4}{2})} \left[\frac{d^2}{dt^2} \int d^{D-1}\mathbf{x}' x'^i x'^j T^{00} \right], \tag{2.15}$$

where $t_{\rm ret} \simeq t - r$, and T^{0m} and T^{lm} were neglected. In four dimensions it is just Eq. (1.39). One can also obtain higher multipole contributions expanding the exponential inside the spacial integral in Eq. (2.12), keeping the terms of consistent order in v. In the next Section we derive the quadrupole-octopole formula keeping the first two terms in this expansion.

The radiated energy in higher dimensional spacetimes can be obtained by the same procedure of Sec. 1.3. Since the gravitational energy-momentum tensor does not depend on the dimensionality of the spacetime [10], the radiated energy in $D-$dimensions is given by

$$\frac{d^2E}{dt d\Omega} = \frac{r^{D-2}}{32\pi} \left\langle \dot{h}_{jk}^{\rm TT} \dot{h}_{jk}^{\rm TT} \right\rangle_{av}, \tag{2.16}$$

and the projector onto the TT gauge in $D-$dimension is [10]

$$\Lambda_{ij,lm}(n) = \delta_{il}\delta_{jm} - n_j n_m \delta_{il} - n_i n_l \delta_{jm} + \frac{1}{D-2} \left(-\delta_{ij}\delta_{lm} + n_l n_m \delta_{ij} + n_i n_j \delta_{lm} \right) + \frac{D-3}{D-2} n_i n_j n_l n_m . \tag{2.17}$$

2.2 Quadrupole-Octopole Formula

The quadrupole approximation is particularly useful since it provides a simple way to compute the radiated energy in the wave zone, for small velocities. The fact that the energy computed in this approximation agrees with other more accurate methods, and its simplicity makes it a valuable tool to estimate gravitational radiation emission. As we shall see in the next Section 2.2 it even provides a fairly good estimate for the radiated energy in processes where it is not always valid. However if one is interested in computing the radiated momentum this formula is not enough, as seen in Sec. 1.4, since the radiated momentum appears in quadrupole-octopole cross terms at lowest order.

Expanding Eq. (2.12) and keeping the first two terms, one gets the quadrupole-octopole formula

$$\bar{h}^{ij}(t,\mathbf{x}) = 4\pi\frac{1}{(2\pi r)^{(D-2)/2}}\partial_t^{(\frac{D-4}{2})}\left[\int d^{D-1}\mathbf{x}'\left(x'^i x'^j\frac{d^2}{dt^2}T^{00} + n_l x'^l x'^i x'^j\frac{d^3}{dt^3}T^{00} - 2n_m x'^i x'^j\frac{d^2}{dt^2}T^{0m}\right)\right], \tag{2.18}$$

where $T^{\mu\nu}$ under an integral means $T^{\mu\nu}(t-r,\mathbf{x})$. In $D=4$ this is Eq. (13) of [3]. Integration by parts casts this equation in a more familiar form (in $D=4$ one gets Bekenstein's Eq. (10) [18]). The last term becomes only dependent on the angular momentum tensor $M^{0ij} = T^{oi}x^j - T^{oj}x^i$, and the two contributions from the octopole order are often called the mass octopole and current quadrupole contributions.
Note that the radiated momentum is obtained integrating the radiated energy, Eq. (1.48), times a factor n^i over the solid angle. Since the integration of an odd number of n_i vanishes, and the projector $\Lambda_{ij,kl}$ has an even number of n_i, the only term which will contribute to the radiated momentum will be the cross term between the quadrupole (no term in n_i) and the octopole (one n_i) contributions to $\bar{h}^{ij}$. For $D=4$ the radiated momentum is equivalent to Eq. (2.19) Ref. [19].
We can now define the momenta of T^{00} and of T^{0i}, which are useful in the following Sections

$$D^{ij}(t) = \int d^{D-1}\mathbf{x}\, x^i\, x^j\, T^{00}(t,\mathbf{x})\,, \tag{2.19a}$$

$$D^{ijk}(t) = \int d^{D-1}\mathbf{x}\, x^i\, x^j x^k\, T^{00}(t,\mathbf{x})\,, \tag{2.19b}$$

$$P^{ij,k}(t) = \int d^{D-1}\mathbf{x}\, x^i\, x^j\, T^{0k}(t,\mathbf{x})\,. \tag{2.19c}$$

Radial infall into a Schwarzschild-Tangherlini black hole

Emission of energy as gravitational waves when a particle falls into a Schwarzschild black hole was one of the first problems to be studied [20, 21], in $D=4$ dimensions. It later served as a model calculation when evolving Einstein equations fully numerically [22, 23]. In Sec. 3.3 this problem is again considered when computing the radiated energy in a black hole collision.
Ruffini and Wheeler [24] first studied the radial infall of a test particle in a Schwarzschild black hole, using a flat-space linearized theory of gravity to compute the radiated energy. The particle's motion was derived from the Schwarzschild metric (in $D=4$). The authors found the total radiated energy to be $0.00246\mu^2/M$, where μ is the point particle's mass, and M is the black hole mass. They also found the energy spectrum to be peaked at a frequency of $0.15/M$. Later Zerilli [20], using the Regge-Wheeler[25] formalism, gave the mathematical foundations for a fully relativistic treatment. Zerilli's equations were then solved numerically for a particle initially at rest at infinity falling into a Schwarzschild black hole by Davis *et al.* [21]. The authors found the radiated energy to be $0.01\mu^2/M$. The spectrum reaches a maximum at $\omega r_H \simeq 0.64$, and then it is exponentially damped, where $r_H = 2M$ is the Schwarzschild radius in $D=4$.

The quadrupole approximation has been used to study this same process in 4 [27] and higher [10] dimensions. For $D=4$ the radiated energy in the quadrupole approximation is $\frac{2}{105}\mu^2/M \simeq 0.019\mu^2/M$, which is of the same order of magnitude as the fully relativistic numerical results. It should be noted that the quadrupole approximation does not hold near the black hole, since the background is no longer flat, and the motion is not slow. Nevertheless, it seems to provide an order of magnitude estimate for the total energy emitted in the process, which means that the radiated energy is dominated by the quadrupole and, in general, by the lowest multipoles. The approximation can also be used to estimate the frequency spectrum of the radiated energy [8]. Since this approximation breaks down somewhere near the horizon, it will only be valid up to a certain t_{max}, which means one does not have the radiated energy for all t to compute the Fourier transform. However when the particle is far from the horizon, $r \gg r_H$, where r is the particle's position, the approximation is valid, and one can estimate the part of the spectrum with $\omega r_H \ll 1$. One also expects the spectrum to be peaked at $\omega r_H \sim 1$, and that the spectrum will be exponentially damped for $\omega r_H \gg 1$, since there is no length-scale smaller than r_H in the problem, which is in reasonable agreement with the numerical results of [21].

Using Eq. (2.18) we can compute the energy radiated away as gravitational waves when a point particle with mass μ falls into a D-dimensional Schwarzschild black hole. The D-dimensional Schwarzschild-Tangherlini metric [28], in spherical coordinates $(t, r, \theta_1, ..., \theta_{D-2})$, is

$$ds^2 = -\left(1 - \frac{16\pi M}{(D-2)\Omega_{D-2}} \frac{1}{r^{D-3}}\right) dt^2 + \left(1 - \frac{16\pi M}{(D-2)\Omega_{D-2}} \frac{1}{r^{D-3}}\right)^{-1} dr^2 + r^{D-2} d\Omega_{D-2}^2 \,. \tag{2.20}$$

Considering a particle falling along a radial geodesic, in the equatorial plane, and at rest at infinity, the geodesic equations give:

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{16\pi M}{(D-2)\Omega_{D-2}} \frac{1}{r^{D-3}} \,, \tag{2.21}$$

If we take the particle to be falling along x^1, one has that $D_{11} = \mu r^2 \left(\frac{dt}{d\tau}\right)^2$, $D_{111} = \mu r^3$ and $P_{11,1} = \mu \frac{dr}{dt} r^2$ are the only nonvanishing components of Eqs. (2.19a) - (2.19c). Here we have made the flat-space approximation, $t = \tau$, in the octopole terms, since the corrections would be higher order terms, this also means that $\frac{dr}{dt}$ can be taken to be given just by Eq. (2.21). However when computing the quadrupole term, one must take these corrections into account since they will give contributions of the same order as the octopole. In addition higher order corrections to the flat space approximation should be taken into account, as it is done in the Post-Newtonian formalism. Nevertheless, we neglect these contribution, since it simplifies matters significantly to consider $t = \tau$, keeping in mind this caveat. Using Eq. (2.18) the only nonvanishing metric component will be

$$\begin{aligned} \bar{h}^{11}(t, \mathbf{x}) &= 4\pi \frac{1}{(2\pi r)^{(D-2)/2}} \left(\partial_t^{\left(\frac{D}{2}\right)} \left(D^{11}\right) + n_1 \partial_t^{\left(\frac{D+2}{2}\right)} \left(D^{111}\right) - 2n_1 \partial_t^{\left(\frac{D}{2}\right)} \left(P^{11,1}\right)\right) \\ &= 4\pi \frac{1}{(2\pi r)^{(D-2)/2}} \left(\partial_t^{\left(\frac{D}{2}\right)} \left(D^{11}\right) + \frac{1}{3} n_1 \partial_t^{\left(\frac{D+2}{2}\right)} \left(D^{111}\right)\right) . \end{aligned} \tag{2.22}$$

Computing the radiated energy, in octopole order, per second and per steradian, we find

$$\frac{d^2E}{dt d\Omega} = 2^{-D+1} \pi^{-(D-3)} \Lambda_{11,11} \left(|\partial_t^{\left(\frac{D+2}{2}\right)} D_{11}|^2 + \frac{1}{9} n_1 n_1 |\partial_t^{\left(\frac{D+4}{2}\right)} D_{111}|^2 + \frac{2}{3} n_1 \left(\partial_t^{\left(\frac{D+4}{2}\right)} (D_{111})\right) \left(\partial_t^{\left(\frac{D+2}{2}\right)} (D_{11})\right)\right) . \tag{2.23}$$

Integration over the solid angle (see Appendix A.1) gives,

$$\frac{dE}{dt} = \frac{2^{2-D} \pi^{\frac{-(D-5)}{2}} (D-3)}{\Gamma[(D-1)/2](D^2-1)} D \left(|\partial_t^{\left(\frac{D+2}{2}\right)} D_{11}|^2 + \frac{1}{9(D+3)} |\partial_t^{\left(\frac{D+4}{2}\right)} D_{111}|^2\right) . \tag{2.24}$$

Note that only the first two terms contribute to the radiated energy, since the integration of an odd number of n_i vanishes.

We now compute the radiated linear momentum in this process, using Eq. (1.67), where S_∞ now denotes a $(N-2)$–sphere at infinity. We have to integrate the radiated energy (2.23) times an extra n_i in order to get the radiated momentum, this means that the only term which will contribute to the radiated momentum is the last term in Eq. (2.23), since it is the only one with an odd number of n_i. Hence we see that the radiated momentum arises from the quadrupole-octopole cross terms. The radiated momentum is then found to be

$$\frac{dP^i}{dt} = \frac{2^{-D+2}\pi^{-(D-5)/2}D(D-3)}{\Gamma\left[(D-1)/2\right](D^2-1)}\frac{2\delta^{i1}}{3(D+3)}\left(\partial_t^{\frac{D+2}{2}}(D_{11})\,\partial_t^{\frac{D+4}{2}}(D_{111})\right). \tag{2.25}$$

In $D = 4$ dimensions this expression for the radiated momentum vanishes, as a consequence of the approximations employed, since we have to take four time derivatives of an octopole moment proportional to t^2, as noted in Ref. [29]. The authors of [29] studied the momentum radiated from a particle falling from rest at infinity along a symmetry axis into a Kerr black hole using a perturbative approach, and found, for a Schwarzschild black hole $\Delta P_z = 8.73\times10^{-4}\mu^2/M$.

Now we can perform the time derivatives and integrate Eqs. (2.24) and (2.25) in time to get the total radiated energy and momentum. The only problem that remains is where to stop the integration. The energy diverges for $r = 0$ but since as the particle approaches the horizon the radiation becomes infinitely red-shifted this should not pose a problem. Furthermore, since the standard [26] picture is that the particle will be frozen near the horizon in the last stages, we can stop the integration at some point near the horizon. We integrate from $r = \infty$ to a point near the horizon, say $r = br_H$, where r_H is the horizon radius and b is a number larger than 1. In Table 2.1 it is shown the energy and momentum computed for several dimensions taking $b = 1$ and $b = 1.2$.

Table 2.1: The energy and momentum, in the quadrupole-octopole approximation, radiated by a particle falling from rest into a higher dimensional Schwarzschild black hole, as a function of dimension. The integration was stopped at a point br_H.

	$\Delta E_{quad}\times\frac{M}{\mu^2}$		$\Delta E_{oct}\times\frac{M}{\mu^2}$		$\lvert\Delta P^1\rvert\times\frac{M}{\mu^2}$	
D	$b=1$	$b=1.2$	$b=1$	$b=1.2$	$b=1$	$b=1.2$
4	0.019	0.01	0	0	0	0
6	0.576	0.049	0.191	0.009	0.220	0.014
8	180	1.19	33.89	0.090	46.95	0.198
10	24567	6.13	41354	2.88	1.8×10^4	2.33
12	3.3×10^6	14.8	1.7×10^7	14.5	3.8×10^7	7.53

We see that in $D = 4$ the radiated energy is only weakly dependent on the cutoff b introduced to stop the integration. This parameter reflects our ignorance of what happens near the horizon, therefore as long as its influence on the radiated energy is small, it probably means the prediction is solid and accurate. However for higher dimensions this is not the case. As the dimensionality of the spacetime grows, so does the difference between the energy and momentum for different values of b, becoming of several orders of magnitude for larger values of D.

Particle in circular orbit

The quadrupole formula has been widely used to estimate the radiation generated by a system of particles orbiting each other, yielding excellent results for orbits with low frequency [30]. Peters and Mathews [31] computed the energy radiated by circular and elliptical orbits in the quadrupole approximation. Its predictions have proved to be consistent with observational data of the binary pulsar PSR B1913+16 [32], since this formalism can account with precision for the increase in period of the pulsar, due to gravitational wave emission [33]. The momentum radiated away in this processes has also been considered by Fitchett in Ref. [27, 35], using the Bekenstein's quadrupole-octopole formalism,

and assuming the motion of the components to be Keplerian. The author also studied the recoil effect on the system due to gravitational wave emission. Later this results we compared against perturbative results for a test particle in a circular geodesic around a Schwarzschild black hole [27, 36]. These results were found to be in very good agreement with the quadrupole-octopole approximation for separations larger than $6M$, where M denotes the black hole mass.
In this Section we consider the motion of two point particles at a fixed distance from each other, and compute the radiated energy and momentum in D-dimensions using the quadrupole-octopole approximation. This procedure can also be applied for elliptic Keplerian orbits (see Refs. [31, 35, 37, 8, 44]). However, since the emission of gravitational radiation tends to circularize the orbits [31, 37, 8], this kind of orbits are relevant in many astrophysical contexts. In fact, as shown in Ref. [8] for a binary of two neutron stars, such as the Hulse-Taylor binary pulsar [32], considering an elliptic Keplerian orbit, the eccentricity goes to zero, to very high accuracy, long before the two neutron stars approach the coalescence phase.
The energy momentum tensor of a system of point particles with masses m_j and velocity v_j is

$$T^{\mu\nu}(t,\mathbf{x}) = \sum_j \gamma_j m_j \frac{dx_j^\mu}{dt}\frac{dx_j^\nu}{dt}\delta^{(D-1)}(\mathbf{x}-\mathbf{x}_j(t))\,, \tag{2.26}$$

where the sum runs over all the particles, $\gamma_j = (1-v_j)^{-1/2}$ is the boost factor and $\mathbf{x}_j(t)$ is the particle's trajectory. For a closed system this is the total energy momentum tensor of the system, and it is conserved. However if external forces act on the system we must take this forces into account when computing the total energy momentum tensor (this is seen in greater detail in Sec. 3.2 also for two particles in a circular orbit). This means that when we plug in the particle's trajectory in Eq. (2.26) that is not a flat-space geodesic, i.e. a straight line, we must take into account the external force that acts on the particle.[1] As we will see in Sec. 3.2, the stresses for this particular system can be though of as the tensions created by infinitely thin massless rod uniting the two particles. Therefore, this stresses only contribute to the purely spacelike components of the energy-momentum tensor, which are not considered in the quadrupole-octopole formalism. This means that we can only consider the particle's contribution to the energy-momentum tensor, since its conservation is imposed when deriving the quadrupole-octopole equations.
Before proceeding we must repeat a caveat similar to the one in the previous Section. For a system bound by gravitational forces, corrections to the flat space approximation should be made if one wanted to expand consistently up to octopole order, since this corrections would be of the same order as the octopole contribution.

Our system consists on two particles of mass m_1 and m_2 at a distance d_1 and d_2 from the origin respectively, rotating around the origin with a rotation frequency Ω. We denote the distance between the two particles by $d = d_1 + d_2$, and place the axes such that the center of mass coincides with the frame's origin. The particle's motion is described by

$$\mathbf{x}_1(t) = (d_1\cos(\Omega t), d_1\sin(\Omega t), 0, \ldots, 0)\,, \tag{2.27a}$$

$$\mathbf{x}_2(t) = (-d_2\cos(\Omega t), -d_2\sin(\Omega t), 0, \ldots, 0)\,. \tag{2.27b}$$

Impose the center of mass to be at the origin gives rise to the following constraint

$$d_1 = \frac{m_2 d}{m_1+m_2}\,, \quad d_2 = \frac{m_1 d}{m_1+m_2}\,, \tag{2.28}$$

and allows us to write the energy-momentum tensor of the system as that of a particle of mass ν with its motion described by the relative coordinate $\mathbf{x}_1(t) - \mathbf{x}_2(t)$. The reduced mass ν is determined by $\nu = \frac{m_1 m_2}{m_1+m_2}$, and we define $\delta = \frac{m_2-m_1}{m_1+m_2}$.

[1] In fact the conservation of the energy-momentum tensor implies that a test particle will move on a geodesic of the background spacetime [38].

We consider the energy momentum tensor to be $T^{00} = \sum_N m_N \delta(\mathbf{x} - \mathbf{x}_N)$ and $T^{0i} = \sum_N m_N \dot{x}^i_N \delta(\mathbf{x} - \mathbf{x}_N)$, regardless of the fact that in order to consistently expand the energy the quadrupole term should have a correction due to higher order terms in the energy-momentum tensor. For the octopole contribution to the energy, and the radiated momentum, however, it suffices to consider this energy-momentum tensor, since any correction would be of higher order. In the above expression the dot denotes a time derivative and the sum runs over the particles, $N = 1, 2$. The nonvanishing momenta D^{ij} are, now written in terms of the reduced mass of the system, ν, given by

$$D^{11} = \nu d^2 \frac{1 + \cos(2\Omega t)}{2}\,, \quad D^{22} = \nu d^2 \frac{1 - \cos(2\Omega t)}{2}\,, \quad D^{12} = D^{21} = \nu d^2 \frac{\sin(2\Omega t)}{2}\,.$$

The momenta D^{ijk} are not simply those of a point particle with mass ν, whose motion is described by the relative coordinate, as there appears an extra factor of δ which vanishes for an equal mass collision.

$$\begin{aligned}
D^{111} &= \nu\delta d^3 \frac{3\cos(\Omega t) + \cos(3\Omega t)}{4}\,, & D^{222} &= \nu\delta d^3 \frac{3\sin(\Omega t) - \sin(3\Omega t)}{4}\,,\\
D^{112} &= D^{121} = D^{211} = \nu\delta d^3 \frac{\sin(\Omega t) + \sin(3\Omega t)}{4}\,, & D^{221} &= D^{122} = D^{212} = \nu\delta d^3 \frac{\cos(\Omega t) - \cos(3\Omega t)}{4}\,.
\end{aligned}$$

Similarly the momenta P^{ijk} are given in terms of the reduced mass ν and the δ factor

$$\begin{aligned}
P^{11,1} &= -\nu\delta d^3 \Omega \frac{\sin(\Omega t) + \sin(3\Omega t)}{4}\,, & P^{22,2} &= \nu\delta d^3 \Omega \frac{\cos(\Omega t) - \cos(3\Omega t)}{4}\,,\\
P^{11,2} &= \nu\delta d^3 \Omega \frac{3\cos(\Omega t) + \cos(3\Omega t)}{4}\,, & P^{22,1} &= -\nu\delta d^3 \Omega \frac{3\sin(\Omega t) - \sin(3\Omega t)}{4}\,,\\
P^{12,1} &= P^{21,1} = -\nu\delta d^3 \Omega \frac{\cos(\Omega t) - \cos(3\Omega t)}{4}\,, & P^{12,2} &= P^{21,2} = \nu\delta d^3 \Omega \frac{\sin(\Omega t) + \sin(3\Omega t)}{4}\,.
\end{aligned}$$

In deriving the above equations we have used the following relations $m_1 d_1^2 + m_2 d_2^2 = \nu d^2$ and $m_1 d_1^3 + m_2 d_2^3 = \nu\delta d^3$. From the previous equations we see that at quadrupole order the energy is radiated at frequency $\omega = 2\Omega$, similarly to what was seen in Sec. 1.4 for a rigidly rotating body. At octopole order we see the energy is radiated at $\omega = \Omega, 3\Omega$, except for an equal mass collision, for which $\delta = 0$ and the metric perturbation at octopole order vanishes. The same radiation resonances are recovered in Sec. 3.2 for a system with two particles in a circular orbit. Since in that Section the energy is not computed in a small velocity approximation we get, not only this first resonances, but also all higher ones, at all multiples of the rotating frequency for an unequal mass system, and at all even multiples of the rotating frequency for an equal mass one. The radiated energy in the quadrupole-octopole approximation is

$$\frac{dE}{dt} = \frac{8D(D-3)}{\pi^{(D-5)/2}\Gamma[(D-1)/2](D+1)(D-2)}\nu^2 d^4 \Omega^{D+2} + \frac{(D-3)D\left(\left(3^{D+2} + 11\right)D + 3^{D+2} + 31\right)d^6\delta^2\nu^2\Omega^{D+4}}{\pi^{(D-5)/2}2^D\Gamma[(D-1)/2](D-2)(D^2-1)(D+3)}\,, \quad (2.32)$$

where the first term is the quadrupole contribution to the radiated energy, which was originally derived in Ref. [10]; and the second is the octopole contribution. The radiated momentum is

$$\frac{dP^i}{dt} = \frac{D(D-3)\left(-3D + 3^{\frac{D+2}{2}}(D+1) - 7\right)d^5\delta\nu^2\Omega^{D+3}}{(D-2)2^{\frac{D+2}{2}}\pi^{\frac{D-5}{2}}\Gamma\left[(D+5)/2\right]}\left(-\sin(\Omega t_{\rm ret})\delta^{i1} + \cos(\Omega t_{\rm ret})\delta^{i2}\right)\,. \quad (2.33)$$

For an equal mass system, that is $m_1 = m_2$, we see that δ vanishes and so does the octopole contribution to energy, as well as the momentum. As pointed out in the introduction, to compute the gravitational energy momentum tensor one must average over several wavelengths. This means that the radiated momentum for a circular orbit over several wavelengths vanishes, as expected by symmetry. However the two black holes will plunge together on an asymmetric trajectory, and there will be a net emission of linear momentum, which makes the newly formed black hole recoil.

The radiated momentum for this system was studied in $D = 4$ dimensions in Ref. [35]. Taking $D = 4$ in Eq. (2.33) we recover their results (Eq. (2.20) of [35]),

$$\frac{dP^i}{dt} = \frac{464}{105}\Omega^7 d^5 \delta\nu^2 \left(-\sin(\Omega t_{\rm ret})\delta^{i1} + \cos(\Omega t_{\rm ret})\delta^{i2}\right)\,. \quad (2.34)$$

The dependence of the radiated momentum on the ratio between the particle's masses is independent of the number of dimensions, being given by $v^2\delta$. Defining $q = m_2/m_1$ we can write the momentum's dependence on the mass ratio q as

$$v^2\delta = (m_1 + m_2)^2 \frac{(1-q)q^2}{(q+1)^5} \,, \tag{2.35}$$

which has a maximum (minimum) for $q = 0.38$ ($q = 2.6$), as found in Ref. [35]. This corresponds to the mass ratio which maximizes the radiated momentum.

Motion of the center of mass

Now we compute the motion of the center of mass, due to the radiation of momentum by the binary system. If we assume that the center of mass is initially at rest, and that its motion is described by

$$(m_1 + m_2)\ddot{\mathbf{r}} = -\frac{d\mathbf{P}}{dt} \,, \tag{2.36}$$

then we have

$$(m_1 + m_2)\ddot{\mathbf{r}} = -\frac{D(D-3)\left(-3D + 3^{\frac{D+2}{2}}(D+1) - 7\right) d^5 \delta v^2 \Omega^{D+3}}{(D-2)2^{\frac{D+2}{2}}\pi^{\frac{D-5}{2}}\Gamma\left[(D+5)/2\right]} \left(-\sin(\Omega t_{\rm ret})\delta^{i1} + \cos(\Omega t_{\rm ret})\delta^{i2}\right) \,. \tag{2.37}$$

Integrating in time, gives

$$(m_1 + m_2)\dot{\mathbf{r}} \;=\; -\frac{D(D-3)\left(-3D + 3^{\frac{D+2}{2}}(D+1) - 7\right) d^5 \delta v^2 \Omega^{D+2}}{(D-2)2^{\frac{D+2}{2}}\pi^{\frac{D-5}{2}}\Gamma\left[(D+5)/2\right]} \left(\cos(\Omega t_{\rm ret}), \sin(\Omega t_{\rm ret}), 0\right) \,, \tag{2.38a}$$

$$(m_1 + m_2)\mathbf{r} \;=\; -\frac{D(D-3)\left(-3D + 3^{\frac{D+2}{2}}(D+1) - 7\right) d^5 \delta v^2 \Omega^{D+1}}{(D-2)2^{\frac{D+2}{2}}\pi^{\frac{D-5}{2}}\Gamma\left[(D+5)/2\right]} \left(\sin(\Omega t_{\rm ret}), -\cos(\Omega t_{\rm ret}), 0\right) \,. \tag{2.38b}$$

Therefore the center of mass moves with speed

$$\frac{D(D-3)\left(-3D + 3^{\frac{D+2}{2}}(D+1) - 7\right) d^5 \delta v^2 \Omega^{D+2}}{(D-2)2^{\frac{D+2}{2}}\pi^{\frac{D-5}{2}}\Gamma\left[(D+5)/2\right](m_1+m_2)} \,, \tag{2.39a}$$

in a circle of radius

$$\frac{D(D-3)\left(-3D + 3^{\frac{D+2}{2}}(D+1) - 7\right) d^5 \delta v^2 \Omega^{D+1}}{(D-2)2^{\frac{D+2}{2}}\pi^{\frac{D-5}{2}}\Gamma\left[(D+5)/2\right](m_1+m_2)} \,. \tag{2.39b}$$

The rotation frequency for a Keplerian orbit is

$$\Omega = \sqrt{\frac{(m_1+m_2)}{d^{D-1}} \frac{8\pi(D-3)}{\Omega_{D-2}(D-2)}} \,, \tag{2.40}$$

which means the center of mass velocity is

$$\frac{\pi(D-3)^{\frac{d+4}{2}} D\left(-3D + 3^{\frac{D+2}{2}}(D+1) - 7\right)\delta v^2}{2^{D+5}(m_1+m_2)^2\Gamma\left[\frac{D-1}{2}\right]\Gamma\left[\frac{D+5}{2}\right]} \left(\frac{r_H}{d}\right)^{\frac{D^2+D-12}{2}} \,. \tag{2.41}$$

We see that the recoil effect increases when the separation of the binary is small, that is of the order of the Schwarzschild radius of the system, $r_H = \left(\frac{16\pi(m_1+m_2)}{(D-2)\Omega_{D-2}}\right)^{1/(D-3)}$. However, this regime is outside the Newtonian limit assumed in the calculation and it can not be relied in this situation. Since $d > r_H$, the suppression due to the last term in Eq. (2.41) is larger for higher dimensions.

In four dimensions this reduces to $\frac{29\delta v^2}{105(m_1+m_2)^2}\left(\frac{r_H}{d}\right)^4$, which is in agreement with Ref. [35]. For small separations, when the recoil effect is larger, this expression can only provide an order of magnitude estimate to the recoil velocity of the final system; and as found by Fitchett in Ref. [27, 35] the recoil speed of the center of mass can be of the order of tens of km/s. There have been other approaches to this problem, using perturbative, Post-Newtonian and numerical techniques (see [39] for a recent review). Numerical results [40] reveal that, when the spin of the body is unimportant, the maximum recoil velocities are of the order of hundreds of km/s. The recoil velocities increase when spin is important [41, 42]. The mass ratio which maximizes the emission of linear momentum is $q = 0.36 \pm 0.03$, which is in good agreement with Ref. [35].

Chapter 3

The Zero Frequency Limit

In the previous chapter we considered an approximation to compute the metric perturbation based on low velocity sources, the multipole expansion. In this chapter we consider a different approach to estimate the radiated energy, which instead of assuming that the velocities involved in the process are low, considers that the process is instantaneous. The advantage of this method is that it depends only on the initial and final states, since for an instantaneous collision the details of the process itself are irrelevant. In addition this method is valid for arbitrarily high velocities, in fact, it provides better estimates to the radiated energy for higher boost factors. This approximation is referred to as the Zero Frequency Limit (ZFL) since it provides a good approximation of the radiated energy at small frequencies. The ZFL method has been widely used, not only for gravitational processes but also for electromagnetical ones [43, 44]. For instance, this method can be used to compute the electromagnetical radiation in the β–decay considering only the initial and final states (see Ref. [43, 44]), and it provides an adequate semi-quantitative description of the radiation.
Wheeler in Ref. [45] also discusses a classical ZFL approach to estimate the emission of gravitational and electromagnetical radiation from impulsive events.
This technique was originally derived from quantum arguments [5, 46, 47, 48], and later from purely classical arguments by Smarr [4]. In this chapter we discuss this technique, starting by briefly reviewing its quantum derivation, and then by presenting Smarr's classical derivation (Sec. 3.1). As an application of this method, we consider the head-on collision (Sec. 3.1) and scattering (Sec. 3.1) of two point particles, as studied by Smarr in Ref. [4]. In Section 3.2 we generalize the ZFL calculation to the case of a non head-on collision between two point particles, and study the radiation spectrum. Finally in Section 3.3 we consider the high-energy collision of two black holes, and study the applicability of the ZFL technique to such process.

3.1 Zero Frequency Limit

The ZFL method lies in the assumption that the collision is instantaneous, so that only the asymptotic states are considered. This technique was derived by Weinberg in 1964 [47, 5] from quantum arguments, but it is equivalent to a purely classical calculation. The idea is to consider a field theory with a spin-2 massless particle (the graviton) propagating in a flat background, and to find the amplitude for the graviton emission in a given process. The gravitational radiation emitted at low frequencies is found by summing the amplitudes for all Feynman diagrams representing the emission of soft gravitons by the collision. One finds that only the gravitons emitted from external lines contribute to the sum, since the contribution from internal lines is negligible when the photon's momentum goes to zero. This is the reason why only the asymptotic states are considered. It should be noted, as pointed out by Weinberg, this method neglects the contributions from soft gravitons emitted from external gravitons lines. This contribution was neglected due to the fact that the effective coupling constant for the emission of a soft graviton from an external graviton line with energy E is proportional to E.

We now review the classical derivation of the Zero Frequency Limit [4]. This approximation assumes that the collision lasts zero seconds and is 'hard', meaning that the incoming and outgoing trajectories asymptotically have constant velocities, at least one of which is nonzero. It is valid for arbitrary velocities, but since we work in a linearized approach the energies have to be low. Furthermore, since this is a long-wavelength approximation, the details of the internal structure of the objects, as well as the details of the collision itself are irrelevant ($\omega^{-1} \gg$ size of the interaction region). Consider the metric perturbation induced at a point x by a point particle. Since we are only interested in the long-wavelength limit, we assume that the point x is very far from the particle, and that the spacetime is nearly flat there. We can now use the linearized approach described in Chapter 1.

We consider spherical coordinates in $\mathbb{R}^3$, where θ is the polar angle and ϕ is the azimuthal angle (see Fig. 3.1). In these coordinates the polarization tensors, for radiation emitted in the direction $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, are

$$\begin{aligned}
\epsilon_+ &= \frac{\sqrt{2}}{2}\left(\mathbf{e}_\theta \otimes \mathbf{e}_\theta - \mathbf{e}_\phi \otimes \mathbf{e}_\phi\right) \\
\epsilon_\times &= \frac{\sqrt{2}}{2}\left(\mathbf{e}_\theta \otimes \mathbf{e}_\phi + \mathbf{e}_\phi \otimes \mathbf{e}_\theta\right) ,
\end{aligned} \tag{3.1}$$

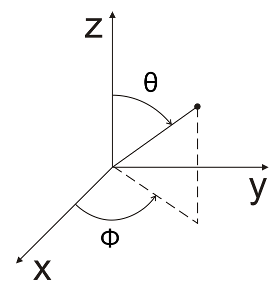

Figure 3.1:

where $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ is an orthonormal reference frame. The energy flux through a 2-sphere in the TT gauge is given by (1.48) or, in terms of these polarization tensors, by:

$$\frac{d^2E}{dtd\Omega} = \frac{r^2}{32\pi}\left\langle\left(\dot{h}_{jk}\epsilon_+^{jk}\right)^2 + \left(\dot{h}_{jk}\epsilon_\times^{jk}\right)^2\right\rangle_{av} . \tag{3.2}$$

We can now define the amplitudes $B_I \equiv \dot{h}_{ij}\epsilon_I^{jk}$, with $I = +, \times$. The Fourier transform of these amplitudes is denoted by $\tilde{B}_I$ [1]. Writing the radiated energy in terms of these amplitudes, and using Parseval's theorem to rewrite it in terms of the Fourier transformed amplitudes one gets

$$\frac{dE}{d\Omega} = \frac{r^2}{32\pi}\sum_I \int_{-\infty}^{+\infty} dt|B_I|^2 = \frac{r^2}{8}\sum_I \int_0^{+\infty} d\omega|\tilde{B}_I|^2 \tag{3.3}$$

where we have used that B_I is real which means that $\tilde{B}_I^*(\omega) = \tilde{B}_I(-\omega)$. Considering now the ZFL one has

$$\frac{d^2E}{d\omega d\Omega} = \frac{r^2}{8}\sum_I \left|\tilde{B}_I(0)\right|^2 , \tag{3.4}$$

where $\tilde{B}_I(0)$ is given by

$$\tilde{B}_I(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt B_I(t) = \frac{1}{2\pi}\left(h_{ij}(t=+\infty)\epsilon_I^{jk} - h_{ij}(t=-\infty)\epsilon_I^{jk}\right) \tag{3.5}$$

Now we can compute the radiated energy per unit frequency per steradian just by considering the metric perturbation at $\pm\infty$, and we see that the ZFL only depends on the asymptotic states. Therefore, there is no need to compute the trajectories of the particles involved in the interaction in this approximation, and the collision is considered to be instantaneous.

[1] Note that Smarr's definition of is different from the one considered here, as he defines $\dot{B}_I \equiv \dot{h}_{ij}\epsilon_I^{jk}$

If we have a particle travelling with 4-velocity v^μ, the metric perturbation can be computed boosting the perturbation caused by the particle in its rest frame. In the particle's rest frame, the trace-reversed perturbation due to the gravitational static field of the particle is

$$\bar{h}_{\mu\nu} = \frac{4M}{r}\delta_{\mu 0}\delta_{\nu 0}\,, \tag{3.6}$$

By a Lorentz boost to the frame where the particle is moving with v^μ we have

$$\bar{h}_{\mu\nu} = \frac{4Mv_\mu v_\nu}{r_{ret}}\,, \tag{3.7}$$

where $r_{ret} = |\mathbf{x} - \mathbf{x}'|$, and $\mathbf{x}$ denotes the point where the field is being computed and $\mathbf{x}'$ the particle's position.
Considering a system of freely moving point particles in the initial and final states, each travelling with 4-momenta $(p^N)^\mu$, where N is the particle index. We consider that each particle causes a perturbation like (3.7), and that the metric is a superposition of such contributions, although this is not always possible, as pointed out in Ref. [50]. If one now plugs this perturbation back in Eq. (3.4) one finds

$$\frac{d^2E}{d\omega d\Omega} = \frac{r^2}{32\pi^2}\sum_I\left(\sum_N \eta_N \frac{4p^N_\mu p^N_\nu}{q\cdot p^N}\epsilon_I^{\mu\nu}\right)^2\,, \tag{3.8}$$

where η_N is ± 1 for particles from the initial and final state respectively, and $q = r_{ret}(1, \mathbf{n})$ and $\mathbf{n}$ is a unit vector pointing from the particle position to the point where the field is being computed. r_{ret} is approximately $r = |\mathbf{x}|$ at large distances from the source, where the field is being evaluated.
Summing over the polarizations this equation is equivalent to Eq. (1.56), if momentum conservation is imposed ($\sum_N p^N = 0$). The advantage of using Eq. (3.8) is that one gets information on the polarization automatically. We can re-write Eq. (3.8) in terms of $k = \omega(1, \mathbf{n})$ instead of q, and if one defines the amplitude for emission of gravitational radiation from this process as

$$a_I = \omega\sum_N \eta_N \frac{p^N_\mu \epsilon_I^{\mu\nu} p^N_\nu}{p^N\cdot k}\,, \tag{3.9}$$

the radiated energy is

$$\frac{d^2E}{d\omega d\Omega} = \frac{1}{2\pi^2}\sum_I |a_I|^2\,. \tag{3.10}$$

Note that a_I is the Feynman-Weinberg-De Witt (FWD) amplitude for the emission of such gravitational radiation from a scattering process.
It should be noted that this technique neglects the loss of energy and momentum by the emission of gravitational waves, so it should not hold if the energy losses through gravitational radiation are large. Even though the method should predict the exact zero frequency limit of the radiation, it is a linearized approach valid only when the radiation is weak. Actually Payne [50] showed that this approximation is not reliable when studying the equal mass head-on collision of two black holes. In fact Payne also compared Smarr's classical derivation with the quantum derivation. As stated above, the emission of soft gravitons by external graviton lines was neglected, however for collisions with a large quantity of radiation present, such as a head-on black hole collision, this emission will make a significant contribution to the ZFL. This is equivalent to Smarr's linearized approach, since that, if gravitational self-interactions are ignored in the spin–2 quantum field theory, the particles in this theory interact exactly as in linearized general relativity.

Using this technique Smarr computed the radiated energy in two different processes, a head-on collision, and a distant encounter. As we shall see, this method provides an estimate to the radiated energy at low frequencies in a head-on black hole collision, being in reasonable agreement with the latest numerical results. However it should be noted that, as pointed out by Payne [50], for the case of a equal mass black hole collision a contradiction arises

when using the ZFL method. The contradiction arises from the fact that the ZFL method predicts a mass for the final hole equal to the total energy of the two colliding particles, which in turn would mean that no gravitational radiation would be emitted. Since these waves carry energy, this implies that $d^2E/(d\omega d\Omega)$ should vanish, thus leading to a contradiction. This means that the ZFL approximation can only be used when the radiation is weak, and one can neglect the radiated energy.
The time-reversed process for a head-on black hole collision had already been computed in 1975 by Adler and Zeks [49]. The authors modelled a supernova explosion as a point particle exploding into two equal mass point particles, moving back to back, and found the same expression for the radiated energy as Smarr (in an equal mass collision).
This method is not expected to give accurate results for small velocities, since in that case the collision would not be instantaneous.

Head-on collision

This collision is modelled by two point particles (the details of the internal structure are irrelevant in this small frequencies approximation) colliding to form a single particle at rest. This means that the radiated energy is computed in the center-of-momentum (CM) frame[2]. The particles are initially travelling along the x–axis, and the final particle is at rest in the origin. The four-momenta of these particles are then given by

$$\begin{aligned} (p_1)^\mu &= \gamma_1 m_1 (1, v_1, 0, 0) \\ (p_2)^\mu &= \gamma_1 m_2 (1, -v_2, 0, 0)\,, \end{aligned} \tag{3.11}$$

with the restriction that $\gamma_1 m_1 v_1 = \gamma_2 m_2 v_2$ (conservation of momentum)[3]. We now proceed to compute the radiated energy in this process. Writing the polarization tensors in terms of $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ and computing the amplitudes from Eq. (3.9) we find

$$a_+ = -\frac{\sqrt{2}}{2}\gamma_1 m_1 v_1 (\cos^2\theta\cos^2\phi - \sin^2\phi)\frac{v_1 + v_2}{(1 - v_1\sin\theta\cos\phi)(1 + v_2\sin\theta\cos\phi)}\,, \tag{3.12a}$$

$$a_\times = \sqrt{2}\gamma_1 m_1 v_1 \cos\theta\cos\phi\sin\phi\frac{v_1 + v_2}{(1 - v_1\sin\theta\cos\phi)(1 + v_2\sin\theta\cos\phi)} \tag{3.12b}$$

And the total radiated energy is found by summing the contribution from each polarization[4], Eq. (3.10) then yields

$$\frac{d^2E}{d\omega d\Omega} = \frac{\gamma_1^2 m_1^2 v_1^2 (1 - \sin^2\theta\cos^2\phi)^2 (v_1 + v_2)^2}{4\pi^2 (1 - v_1\sin\theta\cos\phi)^2 (1 + v_2\sin\theta\cos\phi)^2}\,. \tag{3.13}$$

After a trivial re-definition of angles, this expression is equal to Eq. (2.16) of Ref. [4] (in Ref. [4] the angular variable θ is the angle between the radiation direction and the momenta of the particles. The substitution of their $\cos\theta$ by $\sin\theta\cos\phi$ yields Eq. (3.13)).

We can now consider two limiting cases, the equal mass collision and an extreme mass ratio collision. For the equal mass collision, $m_1 = m_2 \equiv m$ and $v_1 = v_2$, the energy is

$$\frac{d^2E}{d\omega d\Omega} = \frac{m^2\gamma_1^2 v_1^4 \left(1 - \sin^2\theta\cos^2\phi\right)^2}{\pi^2\left(v_1^2\sin^2\theta\cos^2\phi - 1\right)^2}\,. \tag{3.14}$$

[2]The center-of-momentum frame is the frame where $\sum_j \mathbf{p}_j = 0$, where j runs over all the particles of the system.

[3]In the original paper [4] the particles are moving along the z–axis, which will result a difference between the angles and amplitudes here and in Ref. [4].

[4]If we had considered the collision to take place along the z axis as in Ref. [4], the only nonvanishing amplitude would be the one corresponding to the + polarization, and there would be no dependence on the azimuthal angle ϕ, as a consequence of the axisymmetry of the system. However for a collision along another axis, this symmetry no longer coincides with the "special" direction introduced by this particular choice of spherical coordinates in $\mathbb{R}^3$, and this simplification is lost.

Integration over the solid angle gives

$$\frac{dE}{d\omega} = \frac{2\gamma_1^2 m^2}{\pi}\left(2 + (1 - v_1^2)\left(1 - \frac{1}{2v_1}(3 + v_1^2)\log\left(\frac{1 + v_1}{1 - v_1}\right)\right)\right) \tag{3.15}$$

For an extreme mass ratio collision, $m_1 \equiv \mu \ll m_2 \equiv M$, $v_1 \gg v_2$ and $\gamma_1\mu \ll \gamma_2 M$, then

$$\frac{d^2E}{d\omega d\Omega} = \frac{\gamma_1^2\mu^2 v_1^4\left(\sin^2\theta\cos^2\phi - 1\right)^2}{4\pi^2(v_1\sin\theta\cos\phi - 1)^2}\,. \tag{3.16}$$

Integrating over the angular variables we have

$$\frac{dE}{d\omega} = \frac{\gamma_1^2\mu^2}{2\pi v_1}\left(8v_1 - \frac{16}{3}v_1^3 - 4(1 - v_1^2)\log\left(\frac{1 + v_1}{1 - v_1}\right)\right) \tag{3.17}$$

Note that this collision, in the limit when the massive particle M remains unaffected by the collision, is the time-reversed gravitational analogous of the $\beta-$decay process in electromagnetism. In fact, the time-reversed of this extreme mass ratio collision has been considered in Ref. [8]. As pointed out by several authors [4, 8] the angular dependence of the gravitational energy radiated is quite different from that of the electromagnetic energy radiated in the analogous process. In the electromagnetic case the radiated energy is [43]

$$\left.\frac{d^2E}{d\omega d\Omega}\right|_{EM} = \frac{e^2v^2}{4\pi^2}\frac{1 - \sin^2\theta\cos^2\phi}{(1 - v\sin\theta\cos\phi)^2}\,. \tag{3.18}$$

The gravitational energy radiated does not show, in the $v \to 1$ limit, the sharp forward peaking (in the x–direction) that the electromagnetic one shows, so it is not beamed into a small angle in the forward direction. This is a consequence of the graviton having spin–2. If one considers the quantum approach to the ZFL, and sums over the Feynman diagrams for the process, this reason is clearer. The difference arises when summing over the two polarizations of the graviton and the photon. While the polarization tensor for the graviton is a rank–2 tensor, since the graviton has spin–2, the polarization tensor for the photon is a rank–1 tensor , since the photon has spin–1. This means that the amplitude for a given Feynman diagram where a photon with 4-momentum k and polarization $\epsilon_\mu(k)$ is emitted, and the particle has a final 4-momentum p will have a sum over the polarizations given by $\sum_{pol}\epsilon_\mu(k)p^\mu$. Whereas the amplitude for the emission of a graviton it will be $\sum_{pol}\epsilon_{\mu\nu}(k)p^\mu p^\nu$.
Using the same quantum field theoretical approach the common factor in the denominator for both processes is also understood. Considering the same process where a graviton, or photon, with 4-momentum k is emitted, the amplitude will have a propagator proportional to $((k + p)^2 - m^2)^{-1}$, where m is the mass of the particle. Since both the graviton and the photon are massless $k^2 = 0$, and $p^2 = m^2$ the there will be a factor $(1 - v\sin\theta\cos\phi)^{-1}$ on the amplitude, and a factor $(1 - v\sin\theta\cos\phi)^{-2}$ on the emission probability.

The radiated energy, Eq. (3.13), has no dependence on the frequency, therefore in order to obtain the total radiated energy one must introduce a cutoff which is characteristic of the process. This was not entirely unexpected, since the same happened in electromagnetism. If the collision lasts for a time δt, frequencies up to $\omega_{max} \sim 2\pi/\delta t$ are generated. When we consider $\delta t \to 0$ contributions to the radiated energy at arbitrarily high frequencies are introduced, even though they have no physical meaning. Therefore a cutoff frequency must be introduced at $\omega_c \sim 2\pi/\delta t$. Hence the ZFL method provides an approximation of the radiated energy for frequencies smaller than ω_c. This cutoff for a black hole collision is discussed in Sec. 3.3.

For small velocities the radiated energy shows the typical $\sin^4\theta$ quadrupole dependence, characteristic of low-velocity infall. We also see that $\frac{d^2E}{d\omega d\Omega}$ goes to zero when $v_1 \to 0$, although ΔE should not vanish, this shows that the ZFL fails if both the initial and final velocities of the particles are zero, as the gravitational radiation emission is dominated by the details of the collision.

Eq. (3.14) is exactly the same energy found by Adler and Zeks, Eq. (2.12) of [49] (if we take into account the different convention on angles, by an appropriate re-definition of angular variables, as explained after Eq. (3.13)). The authors considered a particle of mass $2M$ at rest blowing up into two point particles, each with energy M, travelling in opposite directions. In the first part of their paper they also considered an infinite acceleration, with the process lasting zero seconds, and computed the metric perturbation and the radiated energy using Eq. (1.52). In the second part of their paper they used a more elaborate model to describe the motion of the two ejecta, using the quadrupole approximation, and found a frequency dependence which justifies the infinite acceleration approach. The spectrum was only flat for small frequencies, and then decayed to zero at large frequencies. Comparison with this small velocities approximations allowed them to estimate the cutoff frequency in this particular process.

The head-on collision of two point particles, in the ZFL approximation, was extended to higher (even) dimensions by Cardoso *et al.* [10]. The authors considered a system of freely moving point particles, whose $(D-1)$-velocities change abruptly at $t = 0$, due to the collision, to form a single body at rest. Using the generalization of Eq. (1.52) to higher dimensions, which they found to be

$$\frac{d^2E}{d\omega d\Omega} = 2\frac{\omega^{D-2}}{(2\pi)^{D-4}}\left(\tilde{T}^{\mu\nu}(\omega,\mathbf{k})\tilde{T}^*_{\mu\nu}(\omega,\mathbf{k}) - \frac{1}{D-2}|\tilde{T}^{\lambda}{}_{\lambda}(\omega,\mathbf{k})|^2\right), \tag{3.19}$$

and plugging the energy-momentum tensor for the system described, the radiated energy is

$$\frac{d^2E}{d\omega d\Omega} = \frac{2}{(2\pi)^{D-2}}\frac{D-3}{D-2}\frac{\gamma_1^2 m_1^2 v_1^2 (v_1+v_2)^2(1-\sin^2\theta_1\sin^2\theta_2)^2}{(1-v_1\sin\theta_1\sin\theta_2)^2(1+v_2\sin\theta_1\sin\theta_2)^2}\times\omega^{D-4}, \tag{3.20}$$

where spherical coordinates in $D-1$ dimensions where considered (see Appendix A)[5]. For $D \neq 4$ the spectrum is no longer flat, but the radiated energy always diverges if it is integrated over all frequencies, therefore a cutoff frequency is required.

Radiated momentum

In this Section we compute the radiated momentum in this process. Although Smarr did not compute the momentum radiated, it can be obtained directly from the radiated energy in the ZFL using Eq. (3.20). We find that it vanishes along the x^i–axis for $i \neq 1$, and for $i = 1$ (the direction of motion) it is

$$\begin{aligned}\frac{dP^1}{d\omega} =& \frac{\gamma_1^2 m_1^2 D/2(D-3)v_1^2\omega^{D-4}}{2^{D-3}\pi^{\frac{D-3}{2}}\Gamma\left[\frac{D+3}{2}\right](v_1+v_2)}\left(\frac{\gamma_1^2 v_1\,{}_2F_1\left[1,\frac{D+2}{2},D+2,\frac{2v_1}{v_1+1}\right]\left(-\left(v_2\left(D+v_1^2\right)\right)+v_1\left(-(D+1)+v_1^2\right)+v_2\right)}{(v_1+1)}-\right.\\ &\left.-\frac{\gamma_2^2 v_2\,{}_2F_1\left[1,\frac{D+2}{2},D+2,\frac{2v_2}{v_2-1}\right]\left(-D(v_1+v_2)-v_1v_2^2+v_1+v_2^3-v_2\right)}{(1-v_2)}+(D+1)\gamma_1^2\gamma_2^2(v_1^2-v_2^2)(1+v_1v_2)\right),\end{aligned} \tag{3.21}$$

where ${}_2F_1$ is the hypergeometric function (recall that D is even). For an equal mass collision the radiated momentum vanishes, as one would expect from symmetry, while for an extreme mass ratio collision the radiated momentum

[5] Note that the authors of [10] have also considered the collision to take place in a different direction. Also the θ_2 in this coordinates is, for $D = 4$ dimensions equal to $\pi/2 - \phi$.

becomes

$$\begin{aligned}
\frac{dP^1}{d\omega} &= \frac{8\,\gamma_1^4\,m_1^2\,v_1^4\,D(D-3)(D/2)!\omega^{D-4}}{\pi^{D/2-1}(1+v_1)(D+1)!}\left(D\,{}_2F_1\left[1,\frac{D+2}{2},D+2,\frac{2v_1}{v_1+1}\right]+\right.\\
&+ \left.(-(D+1)+v_1)\,{}_2F_1\left[1,\frac{D+4}{2},D+3,\frac{2v_1}{v_1+1}\right]+1+v_1\right). \qquad (3.22)
\end{aligned}$$

If we consider the 4-dimensional case, we find that the radiated momentum in the $x^1 = x$ direction is

$$\begin{aligned}
\frac{dP^x}{d\omega} &= \frac{m_2^2}{\pi v_2^2(v_1+v_2)}\left(v_2^2\,(v_1-v_2)+3v_1+5v_2\right)\text{arctanh}v_2 + \frac{m_1^2}{\pi v_1^2(v_1+v_2)}(v_1^2(v_1-v_2)-5v_1-3v_2)\text{arctanh}v_1 + \\
&+ \frac{\gamma_1^2 m_1^2}{\pi v_1 v_2^3}(v_1-v_2)\left(v_1^2\left(v_2^2-3\right)-5v_1v_2-3v_2^2\right), \qquad (3.23)
\end{aligned}$$

whose absolute value has a maximum for (v_1, v_2) equal to $(0.366, 0.133)$ or the other way around, which corresponds to a ratio m_1/m_2 equal to 0.34 or 2.9 respectively. The maximum momentum radiated is then $\sim 0.0031\gamma_1 m_1^2 v_1^2$. For an extreme mass ratio collision the radiated momentum simplifies to

$$\frac{dP^x}{d\omega} = \frac{m_1^2\gamma_1^2\left[v_1\left(15-13v_1^2\right)-3\left(v_1^4-6v_1^2+5\right)\text{arctanh}v_1\right]}{3\pi v_1^2}. \qquad (3.24)$$

Gravitational scattering

Smarr also used the ZFL to compute the energy radiated for the gravitational scatter of a point particle of mass m and velocity v, by a much heavier particle of mass M at rest [4]. This should give an approximation of the energy radiated in the scattering of a small particle (or black hole) by a larger black hole. Consider a small particle initially moving up on the z axis with velocity v, being scattered by a much larger particle, which is fixed on the x axis at $x = b$. For large values of the impact parameter b the scattering angle is

$$\Delta\theta = r_H\frac{1+v^2}{bv^2}, \qquad (3.25)$$

where r_H is the Schwarzschild radius of the large particle. This is valid only in the test particle limit, since the scattering angle is computed from the geodesic equations of the Schwarzschild metric [51]. The initial and final momenta of the small particle are

$$\begin{aligned}
p^i &= \gamma m(1,0,0,v)\\
p^f &= \gamma m(1,v\Delta\theta,0,v) \qquad (3.26)
\end{aligned}$$

Only small angle, large impact parameter scattering is considered. It is required, for a given b and $\delta = v\Delta\theta$, that $\sqrt{r_H/b} < v < 1-\frac{\delta^2}{2}$ and $\theta > \delta v^{-1}$. Then one can compute the FWD amplitudes, and the radiated energy using Eqs. (3.9) and (3.10),

$$\frac{d^2E}{d\omega d\Omega} = \frac{m^2v^2\gamma^2\delta^2}{\pi^2}\left(\frac{\sin^2\theta\cos^2\phi\left(2\cos\theta-v\left(\cos^2\theta+1\right)\right)^2}{4(1-v\cos\theta)^4}+\frac{\sin^2\theta\sin^2\phi}{(1-v\cos\theta)^2}\right). \qquad (3.27)$$

The first contribution to the radiated energy is from the + polarization, and the second from the × polarization. As for the head-on collision there is no dependence on the frequency, which means that the integration over all frequencies be divergent, and a cutoff ω_c is required if one intends to estimate the total energy. Integration over the angular variables yields

$$\frac{dE}{d\omega} = \frac{\gamma^2m^2\delta^2}{\pi}\left(\frac{1}{v^5}\left[2v-\frac{4}{3}v^3-(1-v^2)\log\left(\frac{1+v}{1-v}\right)\right]+\frac{1}{v^3}\left[-4v+2\log\left(\frac{1+v}{1-v}\right)\right]\right). \qquad (3.28)$$

The total radiated energy is then given by $\Delta E = \frac{dE}{d\omega}\omega_c$. Smarr compared the ZFL calculation against different approaches [4], from which the cutoff frequency for this process can be estimated. Note that the integration on the solid angle assumes the cutoff frequency to be independent of the angular variables, which is not necessarily the case. In fact, comparison with the perturbative approach of Peters [52] revealed a disagreement on the velocity dependence of the total radiated energy, which could possibly be explained, as pointed out in Ref. [4], by introducing a cutoff frequency dependent on the direction[53, 54, 55]. This means that one could introduce instead an effective cutoff, "weighted" on the angular coordinates. A similar situation is found when considering a head-on collision, as shown in Sec. 3.3.

We now consider an equal mass scattering in the Newtonian limit, i.e., small velocities and weak gravitational fields. Let us consider the scattering, in the xz plane, of two equal mass particles of mass m and velocity v, in the center-of-momentum frame. We take particle 1 (2) to be at $x = b/2$ ($-x = b/2$) moving in the positive (negative) z direction. Once again we focus only a large impact parameter, small deflection scattering, such that $\delta \ll v$, where $\delta = \frac{m}{vb}$ is the velocity each particle acquires along x (and in opposite directions) after being scattered. The radiated energy is

$$\frac{d^2E}{d\omega d\Omega} = \frac{m^2v^2\delta^2\sin^2(2\theta)\cos^2\phi}{\pi^2\left(v^2\cos^2\theta - 1\right)^4} + \frac{4m^2v^2\delta_1^2\sin^2\theta\sin^2\phi}{\pi^2\left(v^2\cos^2\theta - 1\right)^2}\,, \tag{3.29}$$

where the terms are respectively the "cross" and "plus" polarizations contributions. Integration over the solid angle yields

$$\frac{dE}{d\omega} = \frac{5m^4}{2b^2} = 2.5\frac{m^4}{b^2}\,, \tag{3.30}$$

if we assume the cutoff frequency to be independent of the angular variables.

The radiated energy in these two processes can be compared against the one from the quadrupole approximation [55, 56]. The authors found the radiated energy to be exponentially damped for frequencies much larger than v/b, and given by $\frac{dE}{d\omega} = \frac{32}{5\pi}\frac{m_1^2m_2^2}{b^2} \sim 2.04\frac{m_1^2m_2^2}{b^2}$ for $\omega \ll v/b$, where m_1 and m_2 are the masses of the two particles being scattered. They also computed the total radiated energy, $\Delta E = \frac{37\pi}{15}\frac{m_1^2m_2^2}{b^2}$. Let us first consider the scatter of a test particle by setting $m_1 = m$ and $m_2 = M$. Taking the $v \to 0$ limit of Eq. (3.28) we find $\frac{dE}{d\omega} = \frac{32}{5\pi}\frac{m^2M^2}{b^2}$, which is in agreement with the quadrupolar approximation. Comparison with the total radiated energy of Ref. [55, 56] yields an effective cutoff of $3.8v/b$. Going back to the equal mass Newtonian collision, we take $m_1 = m_2 = m$, and we see that the quadrupole only differs from the ZFL, Eq. (3.30), on the numerical factor. Once again the cutoff will be some undetermined constant times a characteristic frequency of the problem, v/b.

Radiated momentum

Going back to the extreme mass ratio scattering (3.27), we compute the radiated momentum using Eq. (1.67). We find that it vanishes in the x and y directions, while for $i = z$ (the original direction of motion of the small particle) it is

$$\frac{dP^z}{d\omega} = \frac{m^2\gamma^2\delta^2\left(-49v^3 - 3\left(5v^4 - 18v^2 + 5\right)\operatorname{arctanh}v + 15v\right)}{6\pi v^4}\,. \tag{3.31}$$

3.2 ZFL: Non Head-on Collision

In this section we generalize the classic ZFL calculations for head-on collisions [4, 49] to the case of collisions with finite impact parameter. The initial configuration consists of two point particles with mass M_j freely moving toward each other with constant, positive velocity v_j, corresponding to boost factors $\gamma_j = (1 - v_j^2)^{-1/2}$ ($j = 1\,,2$). For convenience the axes are oriented such that the initial motion is in the x–direction (see Fig. 3.2). We assume that at $x = 0$

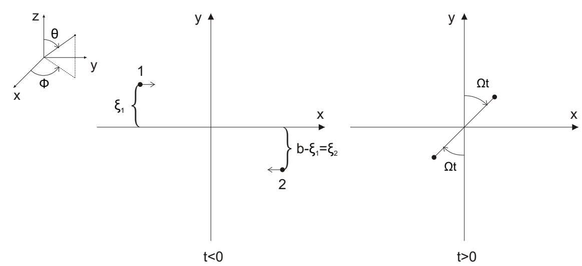

Figure 3.2: The system before and after the collision.

the particles "collide" with generic impact parameter b and form a single final body (strictly speaking this assumption is only valid for small impact parameters, because we expect the bodies to scatter when b is large enough). Angular momentum conservation requires the final body to be rotating. Since the collision is not head-on (and since the energy loss is not included in the motion of point particles), some confining force is necessary to bind the particles. In fact, we show below that additional "stresses" are required to guarantee energy conservation (cf. Ref. [57, 58]). Before the collision and in the laboratory frame the particles have four-positions and four-momenta given by:

$$\begin{aligned}
(x_1)^\mu &= (t, v_1 t, \xi_1, 0)\,, \quad x_2^\mu = (t, -v_2 t, -\xi_2, 0)\,,\\
(p_1)^\mu &= \gamma_1 M_1 (1, v_1, 0, 0)\,, \quad (p_2)^\mu = \gamma_2 M_2 (1, -v_2, 0, 0)\,,
\end{aligned} \tag{3.32}$$

where ξ_1 ($-\xi_2$) is the projection of the position of particle 1 (2) along the y–axis before the collision. If the system's center of mass is at $y = 0$, and b denotes the impact parameter, we have

$$\xi_1 = \frac{b\gamma_2 M_2 (1 + v_{CM} v_2)}{\gamma_1 M_1 (1 - v_{CM} v_1) + \gamma_2 M_2 (1 + v_{CM} v_2)}\,, \quad \xi_2 = (b - \xi_1)\,, \tag{3.33}$$

where v_{CM} is the center-of-momentum frame velocity,

$$v_{CM} = \frac{\gamma_1 M_1 v_1 - \gamma_2 M_2 v_2}{\gamma_1 M_1 + \gamma_2 M_2}\,, \tag{3.34}$$

which corresponds to a boost factor γ_{CM}.

At $t = 0$ the particles become constrained to move as if they were attached to an infinitesimally thin, massless rod of length b. This fictional rod is an idealization but it is necessary to guarantee energy-momentum conservation. For $t > 0$ the particles remain attached to the rod, so that (in the center-of-momentum frame) they rotate around the origin at fixed separation b. Using primes to denote final states, the four-positions and four-momenta after the collision, in the laboratory frame, are

$$\begin{aligned}
(x_1^\mu)' &= (\gamma_{CM}\tau + \gamma_{CM} v_{CM} \xi_1 S, \gamma_{CM} v_{CM}\tau + \gamma \xi_1 S, \xi_1 C, 0)\,,\\
(x_2^\mu)' &= (\gamma_{CM}\tau - \gamma_{CM} v_{CM} \xi_2 S, \gamma_{CM} v_{CM}\tau - \gamma_{CM} \xi_2 S, -\xi_2 C, 0)\,,\\
(p_1^\mu)' &= \gamma_1' M_1 (\gamma_{CM} + \gamma_{CM} v_{CM} \xi_1 \Omega C, \gamma_{CM} v_{CM} + \gamma_{CM} \xi_1 \Omega C, -\xi_1 \Omega S, 0)\,,\\
(p_2^\mu)' &= \gamma_2' M_2 (\gamma_{CM} - \gamma_{CM} v_{CM} \xi_2 \Omega C, \gamma_{CM} v_{CM} - \gamma_{CM} \xi_2 \Omega C, \xi_2 \Omega S, 0)\,,
\end{aligned} \tag{3.35}$$

where $S \equiv \sin(\Omega\tau)$, $C \equiv \cos(\Omega\tau)$, and τ is the time measured in the center-of-momentum frame and primes denote final states. The energy of the particles after the collision (in the center-of-momentum frame) is given by:

$$\gamma_1' M_1 = \gamma_1 M_1 \gamma_{CM} (1 - v_{CM} v_1)\,, \quad \gamma_2' M_2 = \gamma_2 M_2 \gamma_{CM} (1 + v_{CM} v_2)\,. \tag{3.36}$$

For an instantaneous collision the energy-momentum tensor of the system is given by

$$T^{\mu\nu}(t,\mathbf{x}) = \sum_{i=1}^{2} \frac{p_i^\mu p_i^\nu}{E_i} \delta^3(\mathbf{x} - \mathbf{x}_i(t))\theta(-t) + \sum_{i=1}^{2} \frac{(p_i^\mu)'(t)(p_i^\nu)'(t)}{E_i'(t)} \delta^3(\mathbf{x} - \mathbf{x}_i'(t))\theta(t)\,, \tag{3.37}$$

and the angular momentum is,

$$S_3 = \int (x^1 T^{20} - x^2 T^{10}) d^3\mathbf{x} = -\Theta(-t)(\gamma_1 M_1 v_1 \xi_1 + \gamma_2 M_2 v_2 \xi_2) - \Theta(t)(\gamma_1' M_1 \gamma_{CM} \Omega \xi_1^2 + \gamma_2' M_2 \gamma_{CM} \Omega \xi_2^2)\,. \tag{3.38}$$

Angular momentum conservation implies that the rotation frequency is

$$\Omega = \frac{\gamma_1 M_1 v_1 \xi_1 + \gamma_2 M_2 v_2 \xi_2}{\gamma_{CM}(\gamma_1' M_1 \xi_1^2 + \gamma_2' M_2 \xi_2^2)}\,. \tag{3.39}$$

The constraining forces

Henceforth we assume to be in the center-of-momentum frame, so that $v_{CM} = 0$ and $\gamma_{CM} = 1$, in order to simplify our expressions. The process in any other frame which is moving with a constant velocity with respect to the CM frame can be obtained by a Lorentz transformation.
If we naively take the stress-energy tensor of Eq. (3.37) to be the full energy-momentum of the system, we would find that it is not covariantly conserved, i.e. $\nabla_\mu T^{\mu\nu} = 0$ for $\nu = t, z$ but $\nabla_\mu T^{\mu\nu} \neq 0$ for $\nu = x, y$. In fact, one finds

$$\begin{aligned}
\nabla_\mu T^{\mu x} &= -\gamma_1 M_1 \xi_1 \Omega^2 S\,\delta(x - \xi_1 S)\delta(y - \xi_1 C)\delta(z)\Theta(t) + \gamma_2 M_2 \xi_2 \Omega^2 S\,\delta(x + \xi_2 S)\delta(y + \xi_2 C)\delta(z)\Theta(t)\,,\\
\nabla_\mu T^{\mu y} &= -\gamma_1 M_1 \xi_1 \Omega^2 C\delta(x - \xi_1 S)\delta(y - \xi_1 C)\delta(z)\Theta(t) + \gamma_2 M_2 \xi_2 \Omega^2 C\delta(x + \xi_2 S)\delta(y + \xi_2 C)\delta(z)\Theta(t)\,.
\end{aligned} \tag{3.40}$$

Physically, this nonconservation of stress-energy is due to neglecting the energy-momentum associated with the fictitious rod that keeps the particles in circular orbit.

Energy-momentum conservation can be enforced by adding an additional term for each particle that represents this constraining force. The contribution of such forces to the gravitational radiation emitted by a particle in circular orbit was studied by Price and Sandberg [58]. By adding a radial tension $\tau_i(r)$ for each particle and imposing that $\nabla_\mu T^{\mu\nu} = 0$ we get the following contributions to the energy-momentum tensor:

$$\begin{aligned}
T^{xx}_{\text{tens}}(t,\mathbf{x}) &= -S^2 \delta(\cos\theta)\Theta(t) \sum_{j=1}^{2} \tau_j(r)\delta(\phi + \Omega t - \phi_j)\,,\\
T^{yy}_{\text{tens}}(t,\mathbf{x}) &= -C^2 \delta(\cos\theta)\Theta(t) \sum_{j=1}^{2} \tau_j(r)\delta(\phi + \Omega t - \phi_j)\,,\\
T^{xy}_{\text{tens}}(t,\mathbf{x}) &= -S\,C\delta(\cos\theta)\Theta(t) \sum_{j=1}^{2} \tau_j(r)\delta(\phi + \Omega t - \phi_j)\,,
\end{aligned}$$

where $\phi_1 = \pi/2$, $\phi_2 = 3\pi/2$ and

$$\tau_j(r) = \frac{M_j \xi_j \Omega^2 \Theta(\xi_j - r)}{r^2 \sqrt{1 - (\xi_j \Omega)^2}}\,, \quad (j = 1,\, 2)\,. \tag{3.41}$$

Here $r = \sqrt{x^2 + y^2 + z^2}$, θ is the polar angle measured from the positive z-axis, and ϕ is the azimuthal angle in the x–y plane measured from the x-axis (see Fig. 3.2). The factor $\left(1 - (\xi_j \Omega)^2\right)^{-1/2}$ is just the boost factor for particle j in circular motion with angular frequency Ω. The stresses vanish for $b = 0$, as one would expect.

Equal-mass collisions

In this Section we study the equal-mass case $M/2 \equiv M_1 = M_2$. Since we are in the center-of-momentum frame we have that $v_1 = v_2 \equiv v$ and $\gamma_1 = \gamma_2 = \gamma$. In addition, after the "collision," the particles stay on a bound circular orbit with radius $b/2$, and the rotation frequency (3.39) reduces to

$$\Omega = \frac{2v}{b}\,. \tag{3.42}$$

It should be noted that the 4-momentum and 4-position are, in this particular case, given by

$$\begin{aligned}(p_1)'^{\mu} &= \gamma_1 m(1, \frac{b}{2}\Omega\cos(\Omega t), -\frac{b}{2}\Omega\sin(\Omega t), 0)\,, \quad (x_1)'^{\mu} = (t, \frac{b}{2}\sin(\Omega t), \frac{b}{2}\cos(\Omega t), 0)\,,\\ (p_2)'^{\mu} &= \gamma_1 m(1, -\frac{b}{2}\Omega\cos(\Omega t), \frac{b}{2}\Omega\sin(\Omega t), 0)\,, \quad (x_2)'^{\mu} = (t, -\frac{b}{2}\sin(\Omega t), -\frac{b}{2}\cos(\Omega t), 0)\,. \end{aligned} \tag{3.43}$$

The Fourier transform of the energy-momentum tensor (3.37) yields

$$\begin{aligned}\tilde{T}^{\mu\nu}(\omega, \mathbf{k}) &= \frac{p_1^{\mu}p_1^{\nu}}{2\pi i E_1(\omega - v_1 k_x)}e^{-ik_y b/2} + \frac{p_2^{\mu}p_2^{\nu}}{2\pi i E_2(\omega + v_2 k_x)}e^{ik_y b/2} + \sum_{j=1}^{2}\int_{-\infty}^{\infty}\frac{p_j'^{\mu}(t)p_j'^{\nu}(t)}{2\pi E_j'(t)}\exp(i\omega t - i\mathbf{k}\cdot\mathbf{x}_j'(t))\Theta(t)dt + \\ &+ \frac{1}{2\pi}\int d^4x\, T^{\mu\nu}_{\rm tens}(t, \mathbf{x})e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}\,,\end{aligned} \tag{3.44}$$

where $\mathbf{k}$ is the wave vector:

$$k_x = \omega\sin\phi\cos\theta\,,\ k_y = \omega\sin\phi\sin\theta\,,\ k_z = \omega\cos\phi\,. \tag{3.45}$$

We also have

$$e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}_j'(t)} = e^{i\omega t}\exp\left(i\lambda_j\frac{\omega b}{2}\sin\theta\sin(\Omega t + \phi)\right)\,, \tag{3.46}$$

where $j = 1, 2$ is the particle index, $\lambda_1 = -1$ and $\lambda_2 = 1$. If we set $\alpha = \Omega t + \phi$ and $\eta_j = \lambda_j\frac{\omega b}{2}\sin\theta$ the last exponential can be written in terms of Bessel functions of the first kind, using the Jacobi-Anger expansion [59]:

$$e^{i\eta\sin\alpha} = \sum_{n=-\infty}^{n=+\infty} J_n(\eta)e^{in\alpha}\,. \tag{3.47}$$

For large n the Bessel functions satisfy [59]

$$J_n(\eta) \sim \frac{1}{\sqrt{2\pi n}}\left(\frac{e\eta}{2n}\right)^n\,. \tag{3.48}$$

A time-integration introduces an additional factor of $1/n$, so the series converges rapidly for large $|n|$ and we can truncate it at some moderately large value of $n = N$ to get an accurate approximation of the integral [6]. Typically, $N \gtrsim 10$ is sufficient for an accuracy of 1% or better.

The integration of the stresses proceeds in a similar way. After integrating in θ and ϕ, the same Bessel function expansion can be used for the time-integration. The integral of Bessel functions with respect to r can be evaluated using the following identity [59]:

$$\int_0^{\eta} J_\nu(r)dr = 2\sum_{k=0}^{\infty} J_{\nu+2k+1}(\eta), \qquad \mathrm{Re}(\nu) > -1\,. \tag{3.49}$$

[6] Actually, the series should be approximated by summing from $n = n_0 - N$ to $n = n_0 + N$, where n_0 is the value of n which maximizes the absolute value of the terms being summed. After the integration terms of the form $1/(\omega - n\Omega)$ appear. This means that the largest contribution to the sum corresponds to some $n_0 \neq 0$. However it can be checked that $N \gg n_0$ for the range of parameters considered here, so the sum can be taken in a symmetric interval around 0.

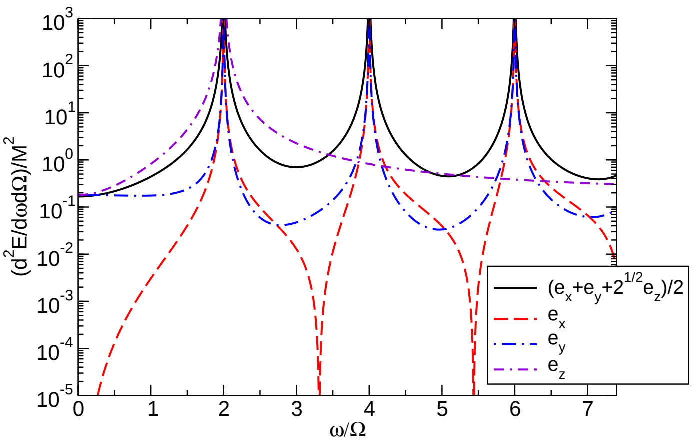

Figure 3.3: Normalized energy per solid angle and per unit frequency emitted in the direction $\hat{\mathbf{k}} = \mathbf{e_x}, \mathbf{e_y}, \mathbf{e_z}, (\mathbf{e_x} + \mathbf{e_y} + \sqrt{2}\mathbf{e_z})/2$ by equal-mass binaries, as a function of ω/Ω.

Radiation spectrum

We now compute the radiated energy for this system using Eq. (1.52). Figure 3.3 shows the energy spectrum along four different directions: $\hat{\mathbf{k}} = \mathbf{e_x}\,, \mathbf{e_y}\,, \mathbf{e_z}\,, (\mathbf{e_x} + \mathbf{e_y} + \sqrt{2}\mathbf{e_z})/2$, where we have set $\gamma = 3$. As long as the energy is plotted as a function of ω/Ω there is no need to specify a value for b, since the energy depends only on the combination $b\omega$. All spectra diverge for $\omega = 2\Omega$, as expected of a rigid symmetric body rotating with angular frequency Ω. For $\hat{\mathbf{k}} = \mathbf{e_z}$ the spectrum only diverges at $\omega = 2\Omega$ (see Ref. [60] for a discussion of particles in circular orbit in the Schwarzschild geometry), but in all other directions the spectrum diverges at even multiples of the rotational frequency Ω, as seen in Secs. 1.4 and 2.2. The same qualitative features hold for higher boost parameters.

Head-on collisions

Let us consider the $b = 0$ limit, which corresponds to a head-on collision and for which we can compare against known results [4, 49]. In this limit, the only nonvanishing components of the energy-momentum tensor are:

$$\begin{aligned}
2\pi\omega\tilde{T}^{tt}(\omega,\mathbf{k}) &= i\gamma M - \frac{i\gamma M}{1 - v^2\sin^2\theta\cos^2\phi}\,,\\
2\pi\omega\tilde{T}^{tx}(\omega,\mathbf{k}) &= -\frac{i\gamma M v^2\sin\theta\cos\phi}{1 - v^2\sin^2\theta\cos^2\phi}\,,\quad 2\pi\omega\tilde{T}^{xx}(\omega,\mathbf{k}) = -\frac{i\gamma M v^2}{1 - v^2\sin^2\theta\cos^2\phi}\,.
\end{aligned}$$

The energy spectrum per unit solid angle is then given by:

$$\frac{d^2E}{d\omega d\Omega} = \frac{\gamma^2 M^2 v^4\left(\sin^2\theta\cos^2\phi - 1\right)^2}{4\pi^2\left(v^2\sin^2\theta\cos^2\phi - 1\right)^2}\,. \tag{3.50}$$

This agrees with previous results in the literature, noting that here M denotes the total mass, therefore $M = 2m$. Indeed it is the same as Eq. (3.14) for a head-on equal mass collision. For ease of comparison with numerical results and with the perturbative calculations (see Sec. 3.3), it is convenient to expand the above expression in spin-weight -2 spherical harmonics. This is explained in Appendix B.

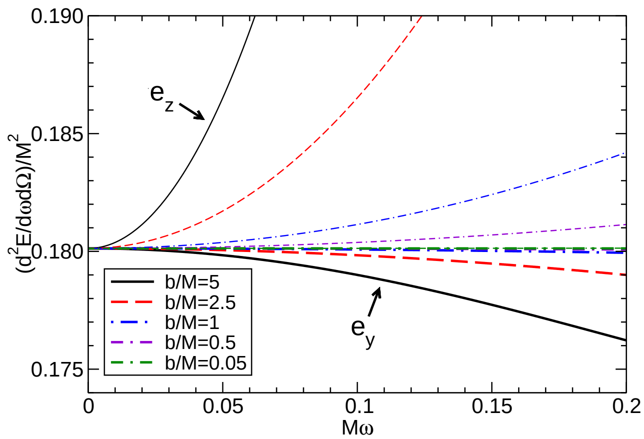

Figure 3.4: Normalized energy spectrum per unit solid angle emitted in the directions $\hat{\mathbf{k}} = \mathbf{e_y}$ (thick lines) and $\hat{\mathbf{k}} = \mathbf{e_z}$ (thin lines) as a function of $M\omega$ for several values of b/M (as indicated in the legend) and $\gamma = 3$. As indicated by Eq. (3.51b) the ZFL for these two different directions is the same and approximately equal to $0.18013M^2$.

Recalling that $\mathbf{e_x}$ corresponds to $(\theta = \pi/2\,, \phi = 0)$, $\mathbf{e_y}$ corresponds to $(\theta = \pi/2\,, \phi = \pi/2)$ and $\mathbf{e_z}$ corresponds to $\theta = 0$, we get, as particular cases

$$\frac{d^2E}{d\omega d\Omega}\Big|_{\omega=0} = 0 \qquad \text{along } \mathbf{e_x}\,, \tag{3.51a}$$

$$\frac{d^2E}{d\omega d\Omega}\Big|_{\omega=0} = \frac{\gamma^2 M^2 v^4}{4\pi^2} \qquad \text{along } \mathbf{e_y},\, \mathbf{e_z}\,. \tag{3.51b}$$

Zero-frequency limit

For arbitrary impact parameters our results show that, in the limit $b\omega \to 0$, the energy spectrum is independent of b and given by Eq. (3.50). This is of course consistent with the head-on results of Smarr [4] and Adler and Zeks [49]. Numerical calculations support this conclusion and reveal additional details for small but nonzero frequencies. The stress terms give the following contributions to the energy-momentum tensor

$$\omega\tilde{T}^{xx}_{\rm tens}(\omega,\mathbf{k})\Big|_{\omega=0} = \omega\tilde{T}^{yy}_{\rm tens}(\omega,\mathbf{k})\Big|_{\omega=0} = -\frac{ib^2\gamma M\Omega^2}{16\pi}\,, \quad \omega\tilde{T}^{xy}_{\rm tens}(\omega,\mathbf{k})\Big|_{\omega=0} = 0\,. \tag{3.52}$$

For $\omega = 0$ the constraining forces provide a nonvanishing contribution to the energy-momentum tensor. It is this particular contribution that allows one to recover the ZFL of the energy spectrum, Eq. (3.50), for *any* impact parameter. This is one of the most intriguing results of this incursion into the properties of the ZFL for collisions with nonzero impact parameter.

The spectra for small frequencies, along the directions $\hat{\mathbf{k}} = \mathbf{e_y}$ and $\hat{\mathbf{k}} = \mathbf{e_z}$ are plotted in Fig. 3.4 for different values of b/M. All spectra have the same ZFL, as discussed above in Eq. (3.51b). Fig. 3.4 refers to $\gamma = 3$, for which we get, from Eq. (3.51b), $\frac{d^2E}{d\omega d\Omega}\Big|_{\omega=0} = 0.18013M^2$ for $\mathbf{e_y}, \mathbf{e_z}$.

For small but finite frequencies, we find that the slope of the energy spectrum depends on the direction and is positive (negative) for $\hat{\mathbf{k}} = \mathbf{e_y}$ ($\mathbf{e_z}$, respectively), and it increases with b/M.

In fact, expanding the energy spectrum for small $b\omega$, up to $O[(b\omega)^2]$, we get:

$$
\begin{aligned}
\frac{d^2E}{d\omega d\Omega} &= \frac{\gamma^2 M^2 v^2}{16\pi^2\left(1-v^2\sin^2\theta\cos^2\phi\right)^2}\Big\{4v^2(\sin^2\theta\cos^2\phi-1)^2 - \\
&- \frac{1}{6}\Big[v^2\left(8v^2+3\right)\sin^6\theta\cos^4\phi+\sin^2\theta\left(\left(8v^2+3\right)\cos 2\phi+12\left(v^2+1\right)\right)- \\
&- \sin^4\theta\cos^2\phi\left(6v^4+\left(2v^2+3\right)v^2\cos 2\phi+20v^2+3\right)-12\Big][b\,\omega]^2\Big\}+O\left[(b\,\omega)^3\right]. \qquad (3.53)
\end{aligned}
$$

Thus, within our model the spectrum typically has quadratic corrections, except along $\hat{\mathbf{k}} = \mathbf{e_x}$, for which the first nonvanishing contribution to the energy spectrum is of order $(b\,\omega)^4$. In fact, we find

$$
\left.\frac{d^2E}{d\omega d\Omega}\right|_{\hat{\mathbf{k}}=\mathbf{e_x}} = -\frac{\gamma^2 M^2\left(3-2v^2\right)^2}{9216\pi^2}(b\,\omega)^4+O\left[(b\,\omega)^5\right]. \qquad (3.54)
$$

This small frequencies behaviour can be seen in Fig. (3.4).

Extreme mass ratio collisions

We now study collisions for $\mu \equiv M_1 \ll M_2 \equiv M$. The energy spectrum can be computed in the center-of-momentum frame. Since particle 2 is much heavier than particle 1, the former is practically at rest in this frame, although we shall not neglect the motion of this particle when we compute the energy-momentum tensor. Therefore, we let $v \equiv v_1 \gg v_2$ and $\gamma \equiv \gamma_1$.

From Eqs. (3.33) and (3.39), the angular frequency and position of particle 1 are given by

$$
\Omega = \frac{\gamma_1\mu v_1\xi_1+\gamma_2 M v_2\xi_2}{\gamma_1\mu\xi_1+\gamma_2 M\xi_2}, \quad \xi_1 = \frac{b\gamma_2 M}{\gamma_1\mu+\gamma_2 M}. \qquad (3.55)
$$

Once again we must add the stresses needed to constrain the particles in their orbits, in order to have a conserved energy-momentum tensor.

Radiation spectrum

The radiated energy is computed using Eq. (1.52) or Eq. (1.54). Once again the Jacobi-Anger expansion (3.47) and Eq. (3.49) are used to compute the Fourier transforms. We expand the energy-momentum tensor in powers of μ/M, and compute the energy keeping only leading-order contributions in μ/M. A calculation of the energy for $\gamma = 3$ in several different directions yields the spectra shown in Fig. 3.5.

The extreme-mass ratio configuration loses the angular symmetry of the equal mass case. Therefore, the spectra now diverge for all multiples of Ω for $\hat{\mathbf{k}} = \mathbf{e_x}$, $\hat{\mathbf{k}} = \mathbf{e_y}$. The behaviour is similar for other directions. This was also found in Sec. 2.2. For $\hat{\mathbf{k}} = \mathbf{e_z}$, the spectrum only diverges for $\omega = 2\Omega$, in agreement with Poisson's findings for particles in circular orbit around black holes [60].

Head-on collisions

For extreme-mass ratio head-on collisions ($b = 0$) we find

$$
\frac{d^2E}{d\omega d\Omega} = \frac{\gamma^2\mu^2 v^4\left(\sin^2\theta\cos^2\phi-1\right)^2}{4\pi^2(v\sin\theta\cos\phi-1)^2}. \qquad (3.56)
$$

This expression coincides, as it should, with Eq. (3.16).
For ease of comparison with numerical and perturbative results of point particles in black hole spacetimes (see Sec. 3.3), in Appendix B we perform a multipole decomposition of this ZFL result in spin-weighted spherical harmonics.

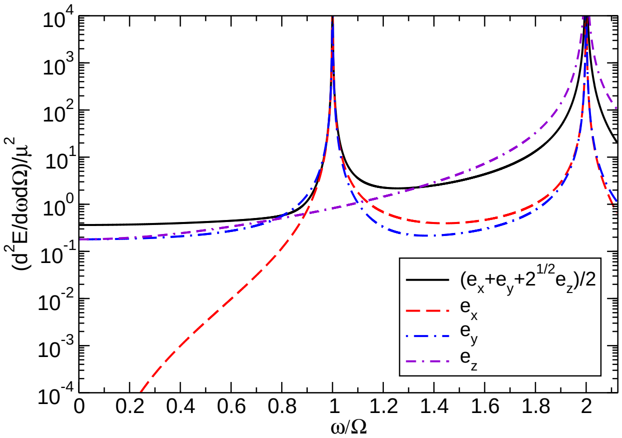

Figure 3.5: Normalized energy spectrum per solid angle emitted in the directions $\hat{\mathbf{k}} = \mathbf{e_x}, \mathbf{e_y}, \mathbf{e_z}, (\mathbf{e_x} + \mathbf{e_y} + \sqrt{2}\mathbf{e_z})/2$ as a function of ω/Ω in the extreme mass ratio case.

Zero-frequency limit

Let us now consider the ZFL for generic values of the impact parameter. As $b\,\omega \to 0$ we find once again that the energy spectrum is independent of the impact parameter (as it was for the equal-mass collisions of section 3.2). The leading-order expression of the energy in powers of μ/M is given by Eq. (3.56), reproducing Smarr's result for head-on collisions. Including higher powers of $b\,\omega$ we get

$$\begin{aligned}
\frac{d^2E}{d\omega d\Omega} &= \frac{\gamma^2\mu^2 v^2}{4\pi^2(v\sin\theta\cos\phi-1)^2}\Big\{v^2(\sin^2\theta\cos^2\phi-1)^2 - \\
&- \frac{1}{192}\Big[(-4v\left(2v^2+3\right)\sin^3\theta(\cos 2\theta+3)\cos 3\phi + 8\sin^2\theta\cos 2\phi\left(\left(3-10v^2\right)\cos 2\theta - 6v^2+9\right) + \\
&+ v\sin\theta\cos\phi\left(\left(372-8v^2\right)\cos 2\theta + \left(6v^2+9\right)\cos 4\theta + 2v^2+387\right) + 8\left(2v^2-21\right)\cos 2\theta + \\
&+ \left(20v^2-6\right)\cos 4\theta - 6\left(6v^2+35\right)\Big][b\,\omega]^2\Big\} + O\left[(b\,\omega)^3\right]. \qquad (3.57)
\end{aligned}$$

As in the equal-mass case, here too the radiation is suppressed along the x–axis, where the leading contribution is of order $(b\,\omega)^4$

$$\left.\frac{d^2E}{d\omega d\Omega}\right|_{\hat{k}=\mathbf{e_x}} = \frac{\gamma^2\mu^2(3-2v(v+3))^2}{576\pi^2}[b\,\omega]^4 + O\left[(b\,\omega)^5\right]. \qquad (3.58)$$

Generic mass ratio

The same procedure can be used to study a collision between two particles of arbitrary masses. We compute the radiated energy, once again in the center-of-momentum frame, for two particles colliding, for $\gamma_1 = 3$ and $M_1 = M_2/3$. The Fourier transforms are once again computed using the Jacobi-Anger expansion and summing only from -10 to 10. In Fig. 3.6 we show the radiated energy per unit frequency per solid angle emitted in several directions. As we can see, similarly to the extreme mass ratio case, all spectra diverge for all multiples of the rotating frequency. The only exception remains $\hat{\mathbf{k}} = \mathbf{e_z}$, where the spectrum only diverges at twice the rotating frequency, as discussed above.
Taking the $\omega \to 0$ limit we recover, once again, the energy obtained by Smarr for the head-on collision, given by Eq. (3.13) (Smarr's Eq. (2.16)).

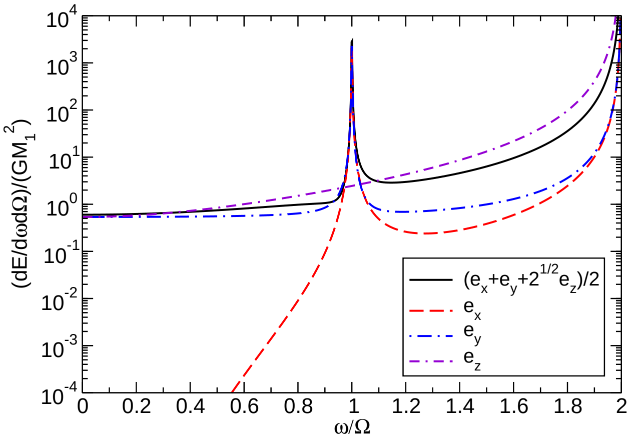

Figure 3.6: Normalized energy per solid angle and per unit frequency emitted in the directions $\hat{\mathbf{k}} = \mathbf{e_x}, \mathbf{e_y}, \mathbf{e_z}, (\mathbf{e_x} + \mathbf{e_y} + \sqrt{2}\mathbf{e_z})/2$ as a function of ω/Ω for mass ratio $q = M_1/M_2 = 1/3$ and $\gamma_1 = 3$. The zero frequency limit agrees with Smarr's energy [Eq. (2.16)] for a head-on collision with unequal mass.

Generality of the model

The most important result of our ZFL calculation for collisions with generic impact parameter is perhaps that the ZFL itself is *independent* of the impact parameter. One of the limitations of the present calculation is that the modelling of the collision is rather ad-hoc, especially in the specification of the nature of the constraining forces. It is natural to ask how the results would change if the constraining forces were modelled differently. For example, in our toy model the final system consists of two particles bound in a circular orbit, so the radiation spectrum shows peaks typical of the radiation produced by rotating bodies. There is a chance that the divergence at harmonics of the rotational frequency of the final system could contaminate the low-frequency behaviour of the spectrum.

To investigate this possibility, instead of considering the collision of two point particles, we studied a point particle colliding with a special extended matter distribution: specifically, we considered an infinitely thin, slowly rotating, uniform disk. For brevity we do not report details of this calculation here. Our main finding is that, if the disk is initially slowly rotating (so that after the collision the system is at rest), the ZFL is the *same* as in the case of two colliding particles. This is by no means a proof that the ZFL is completely independent of the way one models the system. It is however a hint that (as physical intuition would suggest) the ZFL should only depend on the asymptotic momenta of the colliding particles.

3.3 High Energy Collision of Black Holes

The high energy collision of two black holes is one of the most violent process one can have in general relativity. The absence of analytical solutions leaves many unanswered questions about what happens in such collisions. These collisions provide a test to the cosmic censorship conjecture, is it possible to form a naked singularity, or will an event horizon always be formed?

Moreover, the interest in ultrarelativistic collisions increased with the proposal that the Planck scale could be as low as a TeV (thus solving the hierarchy problem) in higher dimensional scenarios, either with large extra dimensions [61] or with compact dimensions with large warp factor [62]. Lowering the Planck scale opens the possibility of black

hole production in particle colliders and ultra high energy cosmic ray interactions with the atmosphere (for a review see [63]). Therefore computing the radiated energy in such processes, as well as the production cross section has a growing interest.
Finally, the AdS/CFT correspondence conjecture [64] further motivates the study of these high-speed black hole collisions. This duality could be used to understand properties of the quark-gluon plasma formed in gold ion collisions at Brookhaven's Relativist Heavy Ion Collider, by studying black hole collisions in AdS [65].

There have been several attempts to understand these ultrarelativistic black hole collisions, by various approaches. A comparison of the techniques described bellow can be found in Ref. [66]. In the seventies Penrose [67] studied the problem of a classical collision with zero impact parameter. Penrose modelled the spacetime as the superposition of two Aichelburg-Sexl waves[7]. When the two shock waves collide, they interact nonlinearly, and a closed trapped surface is formed. By finding this closed trapped surface Penrose was able to set a lower bound on the mass of the final hole of $\sqrt{s/2}$, where $\sqrt{s}$ is the center-of-momentum energy. Later D'Eath and Payne [69, 70, 71] refined this calculation and found an estimate of $0.84\sqrt{s}$ for the mass of the final hole. However this energy was obtained by computing the first two terms of a series expansions of the news functions, and it does not include the radiation emitted during the decay of the final black hole to equilibrium (Schwarzschild). This construction was expanded to non head-on collisions in arbitrary dimensions by Eardley and Giddings [72], and a numerical investigation of their equations was performed in [73, 74].

Another method to compute the radiated energy is to use a perturbation theory approach, considering a small point particle falling into a large black hole. The background metric is taken to be that of the black hole, and the metric perturbation $h_{\mu\nu}$ is induced by the point particle. The Einstein equations can then be expanded to the first order in $h_{\mu\nu}$, and the problem is expressed as a second order differential equation for the metric perturbation. This method was first used to study the head-on collision of two bodies in 1971 in [21], using the Zerilli equation for black hole perturbations [20]. The authors considered a small test particle falling radially, from rest at infinity, into a Schwarzschild black hole. These calculations were also extended to the case of infall from finite radius [75]. Later Ruffini generalized these results for a particle falling with an initial velocity [26]. These results were then extended to a massless particle falling from infinity through a radial geodesic [76], thus extending the previous results for large boost factors. The extension of these results for a spherically-symmetric black hole in D dimensions was also carried out in Ref. [77]. The gravitational radiation for generic orbits of particles falling, from rest at infinity, into a Kerr black hole was studied by Sasaki, Nakamura and collaborators in [78]. These calculations were then extended for a highly relativistic point particle falling into a Kerr black hole, using the Sasaki-Nakamura formalism in [79, 80]. This method should only hold as long as the particle's energy E is much smaller than the black hole's mass M_{BH}. However these results can be extrapolated for energies $E \sim M_{BH}$ and still give sensible results [76, 79, 80]. The Post-Newtonian approximation also provides a way to study these collisions. The radiated energy in the head-on collision of two bodies of arbitrary masses, starting from rest, is computed in [81]. The authors of [81] then compare the Post-Newtonian results with those obtained using perturbation theory.

Another approach was explored by Smarr in 1977 [4], where he used the Zero Frequency Limit approximation to compute the gravitational radiation emitted during the head-on collisions and scattering of black holes, modelled as point particles. Although the total energy depends on a parameter not fixed by the theory, this method provides an estimate of the energy radiated per unit frequency, its polarization and angular distribution. This approach is in excellent

[7] The Aichelburg-Sexl metric [68] corresponds to the Schwarzschild metric boosted, and taking the limit of large boost and small mass, keeping the total energy fixed.

agreement with the perturbative calculations described above [76, 77, 79, 80]. Since this method requires a linearized approximation, it would not be expected to hold for the high-energy collision of two black holes, as pointed out in the previous Sections, which should in principle require the inclusion of nonlinear effects. Surprisingly this method still describes quite well even the nonlinear ultrarelativistic head-on collision of black holes [83]. We now compare the ZFL approximation with perturbative calculations and numerical results, which justifies our interest in such an approximation, providing an additional motivation to the ZFL extension for non head-on collisions, carried out in the previous Section.

Let us consider the extreme mass ratio black hole collision described in Sec. 3.1. For such collisions Smarr proposed that the cutoff frequency ω_c could be the inverse of the large black hole radius $\omega_c \sim r_H^{-1}$, where $r_H = 2M$. As we will see, this cutoff actually corresponds to a "weighted average" of the lowest damped quasi-normal modes.
Taking the limit $v \to 1$ in Eq, (3.16), so that these results can be compared against the perturbative results found in [76] for a ultrarelativistic test particle falling into a Schwarzschild black hole we get

$$\frac{dE_{ZFL}}{d\omega} = \frac{4}{3\pi}\gamma^2\frac{\mu^2}{M} \sim 0.4244\gamma^2\frac{\mu^2}{M}\,. \tag{3.59}$$

Cardoso and Lemos [76] found the radiated energy at zero frequency to be

$$\left.\frac{dE}{d\omega}\right|_{\omega=0} = 0.4244\gamma^2\frac{\mu^2}{M}\,, \tag{3.60}$$

which is in excellent agreement with the ZFL method. If we now integrate Smarr's result up to $\omega_c \sim r_H^{-1}$ to have an estimate for the total radiated energy, we get Smarr's Eq. (3.5)

$$\Delta E_{ZFL} \sim 0.2\gamma^2\frac{\mu^2}{M}\,. \tag{3.61}$$

However this seems to underestimate the total radiated energy, as shown in [76], where the authors found

$$\Delta E \sim 0.262\gamma^2\frac{\mu^2}{M}\,, \tag{3.62}$$

which equivalent to set the cutoff frequency at $\omega_c \sim (1.63M)^{-1}$. For comparison with the perturbative calculations it is convenient to expand this results in spin-weight -2 spherical harmonics, as described in Appendix B. The multipole content of the radiated energy is then given by

$$\frac{dE_{l0}}{d\omega} = \frac{4\gamma^2\mu^2}{\pi}\frac{(2l+1)(l-2)!}{(l+2)!}\,, \tag{3.63}$$

where only the $m = 0$ modes contribute. The perturbative results are in agreement with this calculation, as the spectra is flat up to a certain cutoff, decaying rapidly to zero afterwards. The cutoff is well approximated by the lowest quasinormal mode, for each l. Therefore the cutoff frequency should be given by some "weighted average" of the fundamental gravitational quasinormal modes frequencies, similarly to what was found when considering the gravitational scattering in Sec. 3.1.
The authors also found in Refs. [79, 80] that the radiated energy at zero frequency has the same value independently of the spin of the large black hole, and for particles falling along the equator [79] or along the symmetry axis [80].
For a head-on collision in higher dimensional spacetimes, Eq.(3.20) is, once again, in agreement with the perturbative approach, as shown in [77], where the authors considered a perturbed Schwarzschild-Tangherlini black hole, by an ultrarelativistic point particle falling radially into the black hole.

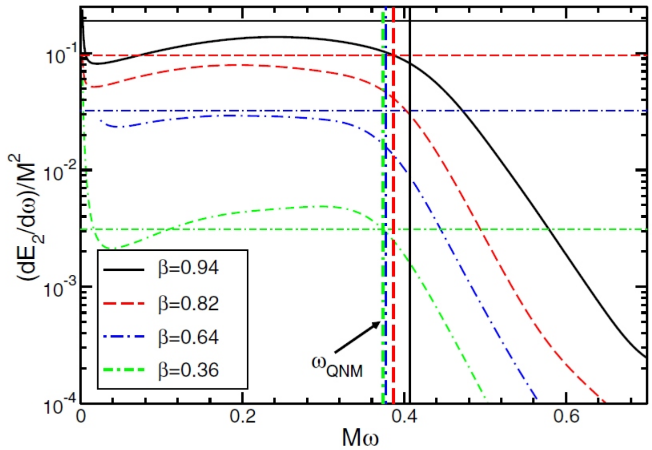

Figure 3.7: Numerical results for the normalized energy spectrum for the $l = 2$, $m = 0$ mode, for several $\beta = v$. Taken from Ref. [83].

For non head-on collisions the perturbative approximation carried out in Ref. [1], for a point particle falling into a Schwarzschild black hole, shows that the zero frequency limit of the radiation spectrum is weakly dependant on the impact parameter, and in quite good agreement with the ZFL prediction (particularly for larger boost factors). In addition Eq. B.13 predicts that, in the ZFL, the ratio between the $l = 2\,, m = 2$ mode and the $l = 2\,, m = 0$ mode is $3/2$, which is in very good agreement with the perturbative results.
The radiated momentum for an extreme mass ratio collision computed in Sec. 3.1 is also in agreement with the perturbative calculations of Ref. [1] for the zero frequency limit of head-on collisions.

So far we have only described analytical methods, which provide estimates and bounds for the radiated energy. We now address the numerical study of these collisions, which is the only way to study black hole collisions with the full nonlinear Einstein equations. The numerical simulation of a black hole collision has been studied for long [22, 82, 83, 84, 85]. What is remarkable is that the simulations of highly boosted head-on black hole collisions is still in reasonable agreement with the ZFL results [83]. Sperhake *et al.* [83] found that, even though the spectrum is no longer flat for small frequencies, it is nearly flat up to a some cutoff frequency, which is well approximated by the least-damped quasinormal mode of the final hole. After the cutoff frequency the spectrum decays exponentially. This can be seen in Fig. 3.7, which is Fig. 2 of Ref. [83]. The $l = 2$, $m = 0$ component of the radiated energy is plotted as a function of $M\omega$, where M denotes the mass of the final black hole, for several values of $\beta = v$. Note that the authors considered the collision along the z-axis, so only the $m = 0$ modes contribute. The straight lines correspond to the ZFL projected into spherical spin-weight harmonics (see Appendix B), and the vertical lines to the least-damped quasinormal mode $l = 2$.

Appendix A

Spherical Coordinates in $(D-1)$−dimensions

In Chapters 2 and 3 spherical coordinates in $(D-1)$−dimensions were considered, when studying gravitational radiation in higher dimensional spacetimes. Here we briefly review some important results. We introduce spherical coordinates $(r, \theta_1, \ldots, \theta_{D-2})$, which are related to the Cartesian coordinates $(x_1, \ldots, x_{D-1})$ by [1]

$$\begin{cases} x_1 = r\sin\theta_1 \sin\theta_2 \ldots \sin\theta_{D-2} \\ x_2 = r\sin\theta_1 \sin\theta_2 \ldots \sin\theta_{D-3}\cos\theta_{D-2} \\ \vdots \\ x_i = r\sin\theta_1 \sin\theta_2 \ldots \sin\theta_{D-i-1}\cos\theta_{D-i} \\ \vdots \\ x_{D-1} = r\cos\theta_1 \end{cases} .$$

This means that the volume element becomes $d^{D-1}\mathbf{x} = r^{D-2} dr d\Omega_{D-2}$, where $d\Omega_{D-2} = \sin^{D-3}\theta_1 \sin^{D-4}\theta_2 \ldots \sin\theta_{D-3} d\theta_1 \ldots d\theta_{D-2}$ is the element of the solid angle. Integration over the angle variables yields $\Omega_{D-2} = \frac{2\pi^{(D-1)/2}}{\Gamma[(D-1)/2]}$. Here, and throughout Chapters 2 and 3, we have used that

$$\int_0^\pi \sin^n\theta = \sqrt{\pi}\frac{\Gamma[(n+1)/2]}{\Gamma[(n+2)/2]} . \tag{A.1}$$

A.1 Useful Integrals

In Chapters 1, 2 and 3 we used some integrals of the projector $\Lambda_{ij,kl}$ (Eq. (2.17)) times a certain number of n^i, over the solid angle, which we list here.

$$\int d\Omega_{D-2} = \Omega_{D-2} \tag{A.2}$$

$$\int d\Omega_{D-2} \underbrace{n^i \ldots n^k}_{\text{odd number}} = 0 \tag{A.3}$$

$$\int d\Omega_{D-2} n^i n^j = \frac{\Omega_{D-2}}{D-1}\delta^{ij} \tag{A.4}$$

$$\int d\Omega_{D-2} n^i n^j n^k n^l = \frac{\Omega_{D-2}}{D^2-1}\left(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right) \tag{A.5}$$

$$\int d\Omega_{D-2} n^i n^j n^k n^l n^m n^o = \frac{\Omega_{D-2}}{(D^2-1)(D+3)}\left(\delta_{ij}\delta_{kl}\delta_{mo} + \ldots\right) , \tag{A.6}$$

where the ... in the final expression denotes all possible pairings.

[1] When using spherical coordinates in $D = 4$ we have that $\phi = \pi/2 - \theta/2$.

Appendix B

Multipolar Decomposition of the Radiated Energy

When considering the radiated gravitational energy in Chapter 3 we performed a multipolar decomposition of the radiated energy, in order to compare the ZFL calculations against known numerical and perturbative results. Multipolar expansions are widely used to study gravitational radiation (see [86] for a review). In numerical relativity it is common to expand in the Newman-Penrose spin-weighted spherical harmonics (SWSH) [87]. In this appendix we review the SWSH, and consider the decomposition of the radiated energy.

B.1 Spin-weighted Spherical Harmonics

Let us consider an orthonormal reference frame $\{X_1, X_2, X_3\}$ such that X_1 and X_2 are tangent to a sphere with radius r, and X_3 is normal to that sphere. Such a frame is defined up to rotations about X_3. If we define $X = \frac{1}{\sqrt{2}}(X_1 + iX_2)$ these rotations can be written as $X' = e^{i\chi}X$. We say that a quantity has spin s if it transforms as $\eta' = e^{is\chi}\eta$ under these rotations. We now choose the frame to be $X_1 = \frac{1}{r}\frac{\partial}{\partial\theta}$, $X_2 = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$, $X_3 = \frac{\partial}{\partial r}$. Let us define the operator ð, acting on a quantity η of spin s, as

$$ð\eta = -(\sin\theta)^s\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right)(\sin\theta)^{-s}\,\eta\,, \tag{B.1}$$

which has spin 1 under rotations.

The spin-weight s spherical harmonics ${}_sY_{lm}$ are then defined by acting with ð on the usual spherical harmonics Y_{lm}

$${}_sY_{lm} = \begin{cases} \sqrt{\frac{(l-s)!}{(l+s)!}}\,ð^s Y_{lm}\,, & 0 \leqslant s \leqslant l \\ \sqrt{\frac{(l+s)!}{(l-s)!}}\,ð^{s*} Y_{lm}\,, & -l \leqslant s \leqslant 0 \end{cases},$$

where the star denotes complex conjugation. Note that the SWSH are not defined for $|s| < l$. One also the following properties

$${}_sY_{lm} = (-1)^{m+s}{}_{-s}Y_{l-m}\,, \qquad {}_sY_{lm}(\pi-\theta,\phi+\phi) = (-1)^l{}_{-s}Y_{l-m}(\theta,\phi)\,. \tag{B.2}$$

The spin-weight spherical harmonics form a complete orthonormal set for each value of s, which means one can expand any quantity of spin s in this basis. The spin-weight $|s| = 2$ is of interest to gravitational radiation [87], and can be obtained from the spin 0 spherical harmonics by

$${}_2Y_{lm} = \sqrt{\frac{(l-2)!}{(l+2)!}}\left(\partial_\theta^2 - \cot\theta\partial_\theta + \frac{m^2}{\sin^2\theta} - \frac{2m}{\sin\theta}\left(\partial_\theta - \cot\theta\right)\right)Y_{lm}\,. \tag{B.3}$$

The usual spherical harmonics Y_{lm} can be determined from the Legendre polynomials $P_{lm}(x)$

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_{lm}(\cos\theta)e^{im\phi}\,. \tag{B.4}$$

B.2 Decomposition of the Radiated Energy

In numerical relativity it is common to expand the Weyl scalar Ψ_4 in spin-weight -2 spherical harmonics. The Weyl tensor is defined as the Riemann tensor with all of its contractions removed $C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} + \frac{1}{3}g_{\rho[\mu}g_{\nu]\sigma}R - g_{\rho[\mu}R_{\nu]\sigma} + g_{\sigma[\mu}R_{\nu]\rho}$, where the square brackets denote antisymmetrization. The contractions of the Weyl tensor with members of the null tetrad are denoted by $\Psi_0\,,\ldots\,,\Psi_4$. As discussed in Sec. 1.2, if we consider a gravitational wave travelling along a direction denoted by $\mathbf{n}$, we can define the "plus" and "cross" polarizations. Let us consider an orthonormal frame $\{X_1, X_2, X_3\}$, with $X_3 = \mathbf{n}$. Then the metric perturbation in this frame defines the "plus" and "cross" components of the metric $h_+ = h^{TT}_{11} = -h^{TT}_{22}$, $h_\times = h^{TT}_{12}$, where the indexes $1,2$ refer to the frame considered. The Weyl scalar Ψ_4 is then given by $\Psi_4 = \frac{1}{2}\left(\ddot{h}_+ - i\ddot{h}_\times\right)$, which is expanded in spin-weight -2 spherical harmonics. The energy radiated through a sphere at infinity, given by Eq. (1.48), can be written in terms of h_+ and $h_\times$ as

$$\frac{d^2E}{dtd\Omega} = \frac{r^2}{16\pi}\left\langle|\dot{h}_+|^2 + |\dot{h}_\times|^2\right\rangle_{av}\,. \tag{B.5}$$

Let us consider the Fourier transform of the radiated energy, which can be written in terms of the Weyl scalar Ψ_4 by

$$\frac{d^2E}{dtd\Omega} = \frac{r^2}{4\pi\omega^2}\left\langle|\Psi_4|^2\right\rangle_{av}\,. \tag{B.6}$$

This means that the radiated energy can also be expanded in spin-weight spherical harmonics in the following way

$$\frac{d^2E}{dtd\Omega} = \left(\sum_{lm}\sqrt{\frac{dE_{lm}}{dt}}\,{}_{-2}Y_{lm}\right)^2\,, \tag{B.7}$$

where $\sqrt{\frac{dE_{lm}}{dt}}$ are yet undetermined functions of t. These functions can be computed using the orthonormality of ${}_{-2}Y_{lm}$ by

$$\sqrt{\frac{dE_{lm}}{dt}} = \int d\Omega\sqrt{\frac{d^2E}{dtd\Omega}}\,{}_{-2}Y_{lm}\,. \tag{B.8}$$

We can now determine the multipolar decomposition of the radiated energy in a head-on collision (Sec. 3.1). Before proceeding, we must go back to the angles chosen in Sec. 3.1, where we considered the collision to be along the x axis. If we had considered a collision along the z axis instead, (as in Refs. [4, 49]) there would be no dependence on the azimuthal angle ϕ, and the calculations when decomposing the energy would be greatly simplified. In what follows we consider the collision to be along the z axis to find the multipolar decomposition of the radiated energy for ultrarelativistic collisions. After obtaining the multipolar decomposition of the energy, a rotation can be performed to change the collision back to the x axis. The multipolar components in a rotated frame can be found following the procedure discussed in Ref. [88].

Equal mass collisions

The radiated energy, for an ultrarelativistic ($v \to 1$) equal mass collision along the x axis is (see Eq. (3.14) and the change of angles discussed above the equation)

$$\frac{d^2E}{d\omega d\Omega} = \frac{m^2\gamma_1^2}{\pi^2}\,. \tag{B.9}$$

Since there is no ϕ dependence only the $m = 0$ modes contribute, and we find

$$\frac{dE_{l0}}{d\omega} = \begin{cases} \frac{16\gamma^2 m^2}{\pi} \frac{(2l+1)(l-2)!}{(l+2)!}, & l \quad \text{even} \\ 0, & l \quad \text{odd} \end{cases},$$

where we have used Eqs. (B.3) and(B.4). We see that the $l = 2$ mode is the one which contributes most, since for large l the energy is suppressed by a factor of $1/l^3$. Summing over even l we find $\frac{dE}{d\omega} = 4\gamma^2 m^2/\pi$, which is in agreement with Eq. (3.15) in the $v \to 1$ limit.

Extreme mass ratio collisions

For an ultrarelativistic, extreme mass ratio collision, along the z axis we find from Eq. (3.16) (changing the angles as discussed above)

$$\frac{d^2E}{d\omega d\Omega} = \frac{\gamma_1^2 \mu^2 (1 + \cos\theta)^2}{4\pi^2}. \tag{B.10}$$

Once again the independence on ϕ means that only the $m = 0$ modes contribute, and we find

$$\frac{dE_{l0}}{d\omega} = \frac{4\gamma^2\mu^2}{\pi} \frac{(2l+1)(l-2)!}{(l+2)!}. \tag{B.11}$$

Summing over l we get $\frac{dE}{d\omega} = 4\gamma^2 m^2/(3\pi)$, which is in agreement with Eq. (3.17), in the $v \to 1$ limit.

Rotation

The two frames considered, one where the collision takes place along the z axis, and the other where it takes place along the x axis, are related by a rotation of $\pi/2$ about the y axis. As shown above, for a collision along the z axis the independence of the ϕ angle means that only the $m = 0$ contribute. However when we rotate back into the previous frame this is not the case. The transformation of the multipolar decomposition of the radiated energy under rotations is explained in Ref. [88]. The multipolar components, for this rotation, transform as follows

$$\sqrt{\frac{dE_{lm}}{d\omega}} = \sqrt{\frac{dE_{l0}}{d\omega}} A^{(l)}_{m0}, \tag{B.12}$$

where we have used the fact that only the $m = 0$ modes contribute, and the matrix $A^{(l)}_{mm'}$ can be found in Ref. [88]. Let us consider the $l = 2$ modes, which are the dominant ones, for the extreme mass ratio collision, in the $v \to 1$ limit. The multipolar components of the radiated energy in the rotated frame (where the collision takes place along x) is

$$\frac{dE_{l\pm2}}{d\omega} = \frac{5}{16}\frac{\gamma^2\mu^2}{\pi}, \qquad \frac{dE_{l0}}{d\omega} = \frac{5}{24}\frac{\gamma^2\mu^2}{\pi}, \tag{B.13}$$

with the $m = \pm1$ modes vanishing. For an equal mass collision, in the same limit, we have that the $m = \pm1$ vanish, and that

$$\frac{dE_{l\pm2}}{d\omega} = \begin{cases} \frac{5}{4}\frac{\gamma^2 m^2}{\pi}, & l \quad \text{even} \\ 0, & l \quad \text{odd} \end{cases}, \qquad \frac{dE_{l0}}{d\omega} = \begin{cases} \frac{5}{6}\frac{\gamma^2 m^2}{\pi}, & l \quad \text{even} \\ 0, & l \quad \text{odd} \end{cases}.$$

For arbitrary velocities one finds for the dominant mode, $l = 2$, and for $m = 0$

$$\frac{dE_{20}}{d\omega} = \frac{5m^2\gamma^2 \left(-10v^3 + \frac{3}{\gamma^4}\log\left(\frac{2}{v+1} - 1\right) + 6v\right)^2}{96\pi v^6}, \tag{B.14}$$

where the rotation has already been performed, such that the collision takes place along the x axis. Similarly for an extreme mass ratio collision we have that

$$\frac{dE_{20}}{d\omega} = \frac{5m^2\gamma^2 \left(v\left(5v^2 - 3\right) + \frac{3}{\gamma^4}\tanh^{-1}(v)\right)^2}{96\pi v^6}. \tag{B.15}$$

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